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A New Pressure-Sinkage Model for Small, Rigid Wheels on Deformable Terrains

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Abstract. *In the 1960s, Bekker developed the formulae that constitute the core of terramechanics. These semi-empirically derived equations allow the designers of large vehicles to understand and predict vehicle mobility performance over deformable terrains. In recent years, researchers have questioned the accuracy with which Bekker theory can be applied to smaller vehicles. Specifically, Bekker theory is known to yield modeling errors for vehicles with wheel diameters and normal loading lower than 50 cm and 45 N, respectively. This paper presents a new pressure-sinkage model to account for these errors. The model is validated against results from 160 tests performed using five wheel diameters and three terrains. It is shown to improve sinkage and compaction resistance modeling accuracy by up to 3.6x and 10x, respectively.*

Keywords. Small wheels, sinkage, compaction resistance, Bekker theory.

1 Introduction

For decades, Bekker theory has been successfully used to model large vehicle mobility performance. In recent years, however, the accuracy of Bekker theory when applied to smaller vehicles has been called into question (Carrier, 1996, Richter et al., 2002, Scott et al., 2008, Meirion-Griffith et al., 2010). Specifically, Bekker theory is known to yield inaccurate models for vehicles using wheels of diameter and normal loading lower than 50 cm and 45 N. This is problematic because these bounds encompass the majority of modern unmanned ground vehicle (UGV) designs. Despite this, very little research has been performed to improve Bekker's model in this regard.

This paper presents a new pressure-sinkage model for small wheels. Sinkage is fundamental, as it appears in the calculation of almost all other mobility metrics. Sections 1 through 3 of this paper provide a review of the relevant theory, sources of error, and an experimental characterization of these errors. Sections 4 and 5 detail the results of an empirical investigation into the effect of wheel diameter on the pressure-sinkage relationship. These results are then used in the derivation of a proposed model for wheel sinkage and compaction resistance. This proposed model is shown to yield significantly more accurate predictions (up to an order of magnitude) than those obtained using existing theory.

1.1 Background

The semi-empirically derived formulae of terramechanics are largely based on geotechnical engineering. Bekker took theoretical understanding from such researchers as Terzaghi,

Mohr and Coulomb and mapped it to the wheel/track-soil interface. The result was a model of the stresses, forces and displacements encountered on deformable terrains. During the conception of his model, Bekker utilized four soil properties pertinent to this paper:

1. Cohesion, C , is the apparent attraction between soil particulates. Analytically, it is the shear strength of a soil under zero normal stress.
2. Internal friction angle, Φ , is a measure of a soil's ability to withstand shear. It is calculated as the angle between a normally applied load and the resultant force vector when shear failure occurs.
3. The sinkage modulus, k , is a measure of a soil's resistance to deformation under normal loading. Its origins lie in Bernstein-Goriatchkin theory, which will be discussed in detail in Section 3.4. Bekker expanded k into three constituent parts, k_c (*cohesive*), k_ϕ (*frictional*), and b (*wheel width*). He also claimed that there existed a linear relationship between k_c and b such that: $k = \frac{k_c}{b} + k_\phi$.
4. The sinkage exponent, n , is a property that affects the curvature of the pressure-sinkage curve for a soil under normal loading.

Bekker also introduced several vehicle performance metrics, four of which are relevant to this paper.

1. Sinkage, z_0 , is the geometric measure of the vertical distance between a soil's undeformed surface and the point located directly beneath the centroid of the wheel (see Figure 2).
2. Compaction resistance, R_c , is the force generated while towing an undriven wheel and is equivalent to the vertical work done on a soil by a normally loaded wheel. It is, in general, the largest factor in motion resistance.
3. Thrust, H , is a measure of the positive tractive force a wheel can generate on a given soil. It is equivalent to the shear stress the terrain at the wheel-soil interface can accommodate before complete shear failure occurs. Bekker calculated the ideal thrust, H_0 , directly from Mohr-Coulomb theory:

$$H_0 = AC + W \tan \phi \quad (1)$$

where A is the wheel-soil contact area and W is the wheel normal load. The concept of thrust may be furthered to include wheel slip, given by Bekker as,

$$H = H_0 \left(1 - e^{-il/\kappa}\right) \quad (2)$$

where i is wheel slip, l is the length of the wheel-soil contact patch and κ is the modulus of deformation.

4. Drawbar pull, DP , is the primary metric of Bekker theory and provides an assessment of a wheel's ability to generate traction. Drawbar pull is the difference between available thrust and resistances.

2 Experimental Evaluation of Sinkage Prediction Accuracy

In order to evaluate the accuracy with which Bekker's equation can predict small wheel sinkage, experiments were performed using a single-wheel vehicle-terrain testbed. This testbed, shown in Figure 1, is instrumented to accurately measure wheel speed, slip, loading, drawbar pull, terrain resistance, and sinkage.

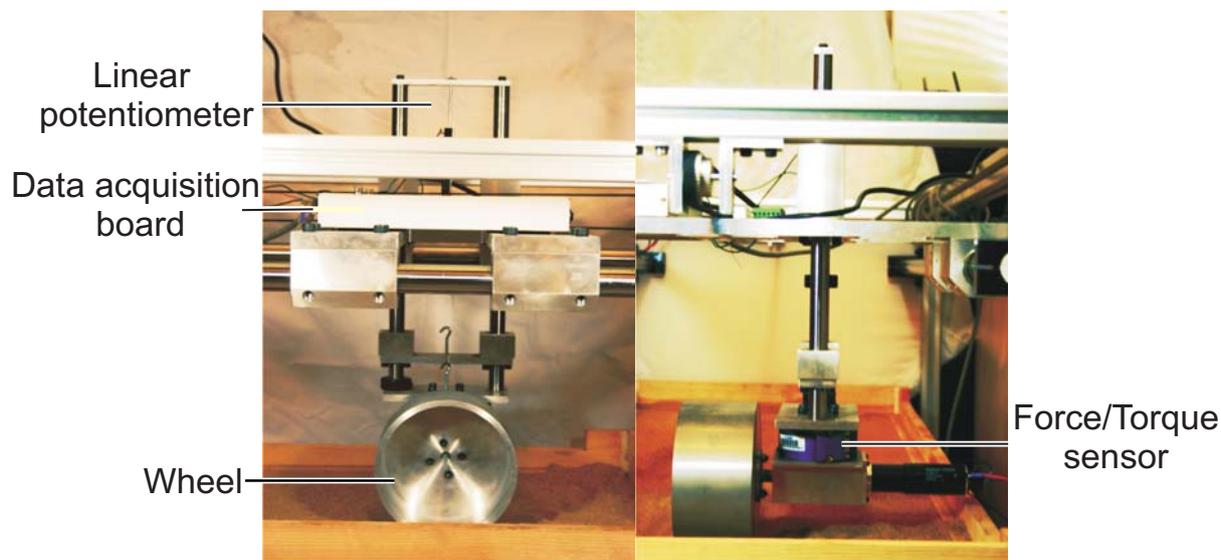


Figure 1: Single-wheel vehicle-terrain testbed

Tests were performed under the conditions given in Table 1, to simulate realistic conditions.

Table 1: Experimental test conditions

Wheel diameter (m)	Wheel width (m)	Speed (cm/s)	Normal loading (N)
0.17	0.085	10	46 and 84

The wheel was driven with zero slip to ensure that the measured sinkage was not due to rutting. The physical properties of the sand, given in Table 2, were found from plate sinkage and direct shear tests.

Table 2: Dry sand physical properties

Cohesion, C (kPa)	Friction angle, Φ ($^{\circ}$)	Sinkage exponent, n	Sinkage modulus, k (kN/m^{n+2})
2.05	27.5	0.75	1783

Following multiple tests at each normal load, predicted sinkage values were found to be significantly below those observed experimentally. Table 3 summarizes these results.

Table 3: Errors in predicted sinkage for 0.17 m diameter wheel (Meirion-Griffith and Spenko, 2010)

Load (N)	Sinkage (z_0) % error
46	33.0
84	43.3

Perhaps more noteworthy than the raw data, is that these errors yield sinkage under-predictions. This means that in the field, small wheels may exhibit greater sinkage than predicted, leading to greater resistances and ultimately, impaired tractive performance.

3 The Sinkage Equation

This section discusses the sinkage equation, its limitations and core assumptions. The purpose of this section is to provide a comprehensive understanding of the sinkage equation, upon which further analysis is built.

3.1 Derivation of the Sinkage Equation

In order to begin our review of Bekker's sinkage equation, the key components of its derivation are presented.

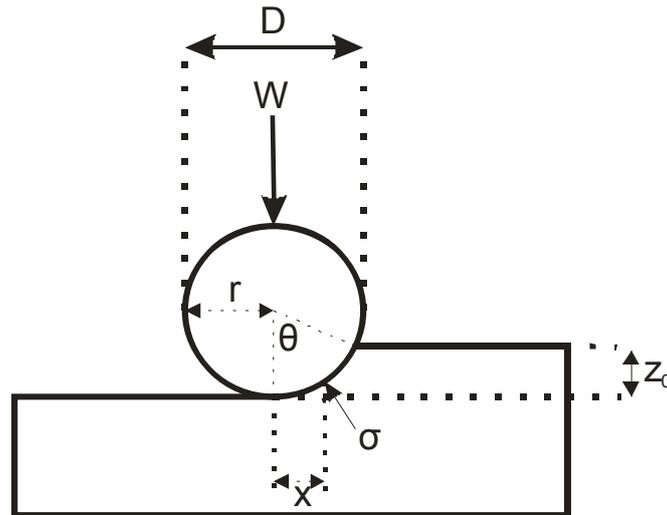


Figure 2: Geometry of the wheel-soil contact area

Bekker began his derivation of the sinkage equation with an expression for W , the normal loading on a wheel:

$$W = b \int_0^{\theta} \sigma r \cos \theta d\theta \quad (3)$$

where σ is the normal pressure over the contact patch and r is the wheel radius.

Similarly, Bekker stated:

$$W = -b \int_0^{\theta} P dx = -b \int_0^{\theta} \left(\frac{k_c}{b} + k_\phi \right) z_0^n dx \quad (4)$$

where P is the pressure and the integrand of (3) has been replaced in (4) by use of Bekker's modified Bernstein-Goriatchkin equation. The Bernstein-Goriatchkin equation relates the pressure, P , on a flat plate, with its subsequent sinkage, z_0 :

$$P = \left(\frac{k_c}{b} + k_\phi \right) z_0^n \quad (5)$$

Solving (4), we obtain:

$$W = \frac{b \left(\frac{k_c}{b} + k_\phi \right) \sqrt{z_0 D}}{3} z_0^n (3 - n) \quad (6)$$

which, after rearranging for sinkage, becomes:

$$z_0 = \frac{3W}{b(3-n) \left(\frac{k_c}{b} + k_\phi \right) \sqrt{D}}^{2/2n+1} \quad (7)$$

The following subsections detail three key assumptions made during the derivation of Bekker's sinkage equation.

3.2 Assumption 1: Sinkage Moduli

In 2002, Richter et al. published results regarding the accuracy of small wheel sinkage predictions as a function of normal load. Figure 3 shows the dependence of the sinkage modulus, k , on normal load. According to Bekker's definition of k , there should be no dependence.

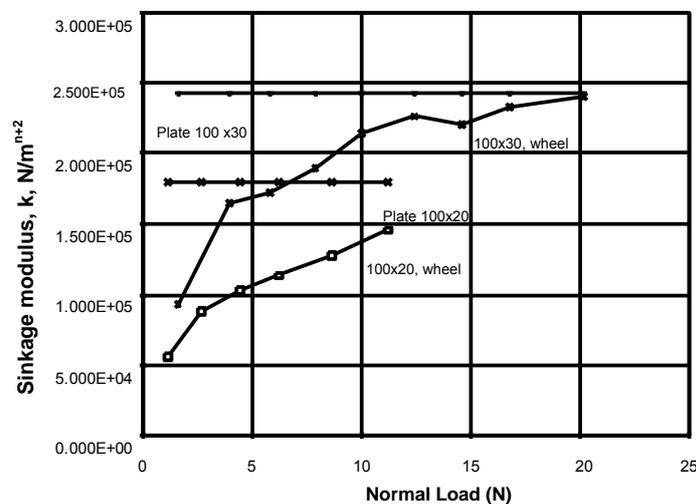


Figure 3: Sinkage modulus, k , for a 100 mm diameter wheel of widths 20 and 20 mm. Reproduced with permission of L. Richter (Richter et al., 2002)

Clearly, as the normal load is increased, the value of k does not remain constant, as required by the sinkage equation.

Findings by Upadhyaya (Upadhyaya, 1993) and Lyasko (Lyasko, 2010) have highlighted further deficiencies in Bekker's sinkage moduli. In these findings, errors were not associated with k directly, but with the manner in which Bekker split k into k_c and k_ϕ . As stated in Section 1, these moduli assume a linear relationship between k_c and the wheel width, b . Data presented by Lyasko showed a weak relationship between b and k , with an R^2 of only 0.1363 (Lyasko, 2010). As such, the separation of k into Bekker's moduli has been not been used for any research presented in this paper.

3.3 Assumption 2: Distribution of Stresses at the Wheel-Soil Interface

Within the derivation of Bekker's equation lie two assumptions regarding the distribution of pressure at the wheel-soil interface. First, the integrand of (3) describes the 2D distribution of normal pressure along the arc length of the wheel-soil contact region. This integrand is then multiplied by the wheel width, b , to yield the pressure distribution over the 3D contact patch. However, this simple multiplication by b requires the assumption that the pressure at the wheel-soil interface is homogeneous across the width of the wheel. From contact mechanics, we know this to be unlikely.

The second assumption lies in the calculation of the soil region bearing the wheel's normal load. Using current theory, we transform the wheel normal load into a pressure by dividing by the area of the wheel-soil contact patch. Figure 4 shows a plastically deformable surface being indented by a normally loaded plate. The log spiral curves show the pressure distribution lines emanating away from the plate and encapsulating an area known as the Rankine zone. The dashed column, located directly beneath the plate, shows the region in which all stresses are assumed to be distributed in Bekker's sinkage equation. Clearly, the pressure-bearing region of the soil is much larger than assumed.

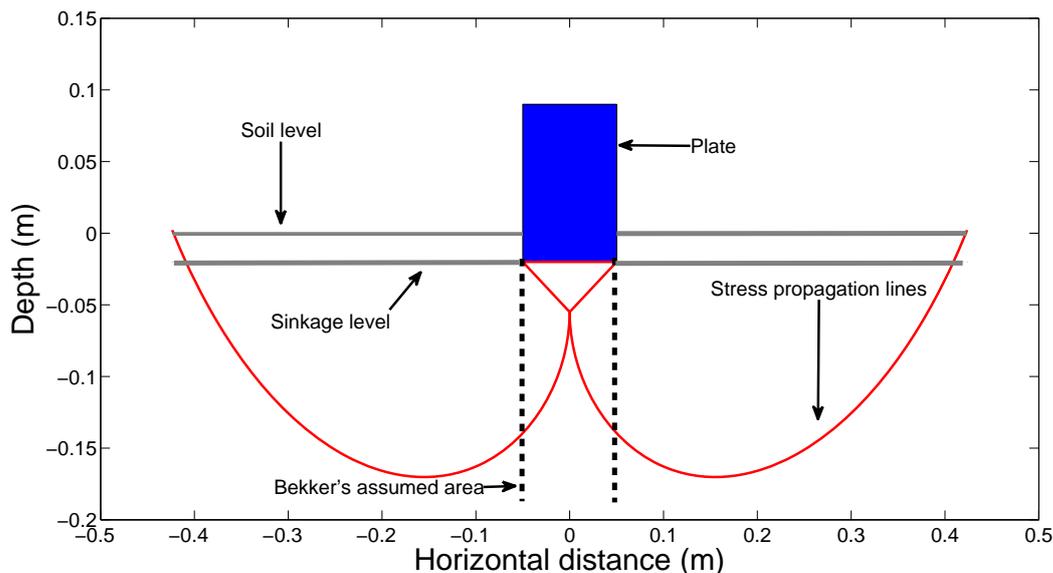


Figure 4: Stress distribution beneath a normally loaded plate on a deformable surface

3.4 Assumption 3: The Flat Plate Approximation

In (7) and (8), we see wheel sinkage given only as a function of terrain constants and applied pressure. As such, Bekker's sinkage equation postulates that there is no dependence on wheel diameter. In Bekker's 1969 publication, *Introduction to Terrain-Vehicle Systems*, he made the following statement:

"Predictions for wheels smaller than 20 inches in diameter become less accurate as wheel diameter decreases, because the sharp curvature of the loading area was neither considered in its entirety nor is it reflected in bevameter tests" (Bekker, 1969).

In (4) we see the substitution of the analytic integrand for the Bernstein-Goriatchkin equation. Negating Bekker's separation of the sinkage moduli, the Bernstein-Goriatchkin equation is given as:

$$P = kz^n \quad (8)$$

This empirically derived expression relates the pressure and subsequent sinkage of a flat plate (rectangular or circular) into a deformable surface. Bekker used (8) in his derivation of the sinkage equation, stating that the flat plate pressure-sinkage relationship was a valid approximation for wheel sinkage. This is a reasonable approximation for large diameter wheels experiencing only modest sinkage, for which the contact patch is indeed rectangular and relatively flat. Note that the form of (8) is a power curve, with k and n being curve fitting constants. These constants are defined per soil, examples of which are given in Table 4.

Table 4: Examples of pressure-Sinkage constants (Wong, 2008)

	Sandy loam	Heavy clay	Snow	Dry sand
Sinkage modulus, k (kN/m ⁿ⁺²)	1515	1556	197	1528
Sinkage exponent, n	0.7	0.13	1.6	1.1

Investigating the pressure-sinkage dependence on wheel diameter for small wheels is the focus Sections 4 and 5.

4 The Pressure-Sinkage Relationship

This section details experimental results that demonstrate the need for a pressure-sinkage model that includes both sinkage and wheel diameter, such that $P = f(z, D)$. Here, we hypothesize that the Bernstein-Goriatchkin expression is insufficient for modeling small wheel sinkage, as it assumes a constant contact area between the wheel and soil. For the sharp curvature of small diameter wheels, this cannot be true. As a wheel sinks from surface level to z_0 , the contact area increases, thus tending to reduce the pressure.

4.1 Experimental Procedure

In order to evaluate the effect of wheel diameter on the pressure-sinkage relationship, tests were conducted using multiple wheel diameters and soils. In total, 160 pressure-sinkage tests were performed. Five wheel diameters were used, ranging from 0.1 m to 0.3 m. This range encapsulates the majority of wheels used on modern UGVs. The physical properties of the three soils chosen for testing are detailed in Table 5.

Table 5: Physical properties of soils used in pressure-sinkage testing

Property	Dry Sand	Calcium Silicate	Earth
Cohesion, C (kPa)	2.05	3.08	3.13
Friction angle, Φ ($^\circ$)	27.2	32.2	34.0

To perform the experiments, a custom-built pressure-sinkage testbed (shown in Figure 5) was utilized.

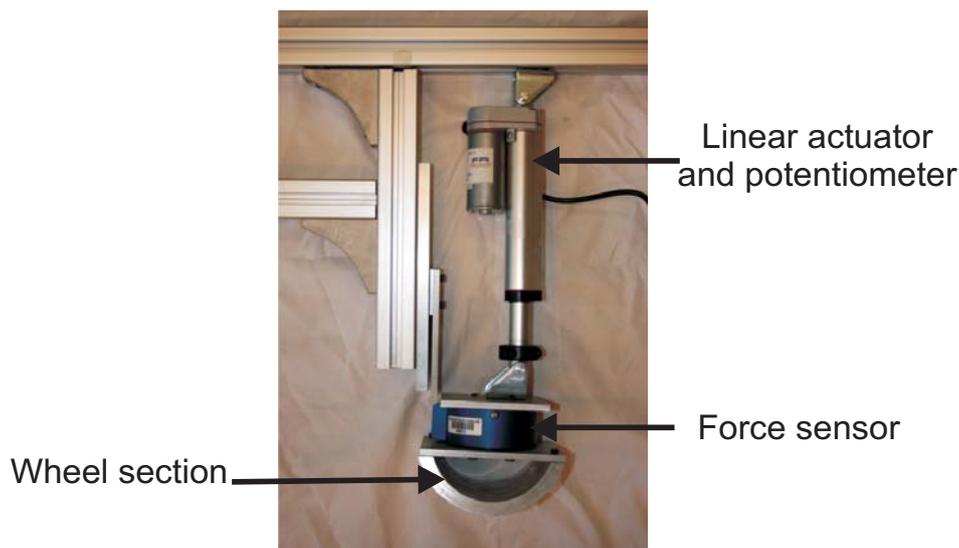


Figure 5: Pressure-Sinkage testbed

The testbed is instrumented with a 6-axis force/torque sensor, linear actuator and potentiometer. The linear actuator provides up to 50 mm of sinkage. The normal load was limited to < 450 N. Note that the low normal loading condition discussed in Section 3.2 is a separate area of study from the tests detailed here; we are only concerned with small wheel diameters. Force and sinkage were measured using the force/torque sensor and linear potentiometer. Required soil bin dimensions were calculated using Terzaghi's bearing capacity theory for shallow foundations. This is necessary to ensure that the soil bin walls do not interfere with the stress distributions beneath the loaded wheel section.

5 Results

Figures 6, 7 and 8 show the data for each wheel diameter and soil. Each figure represents one soil, within which each line represents one diameter.

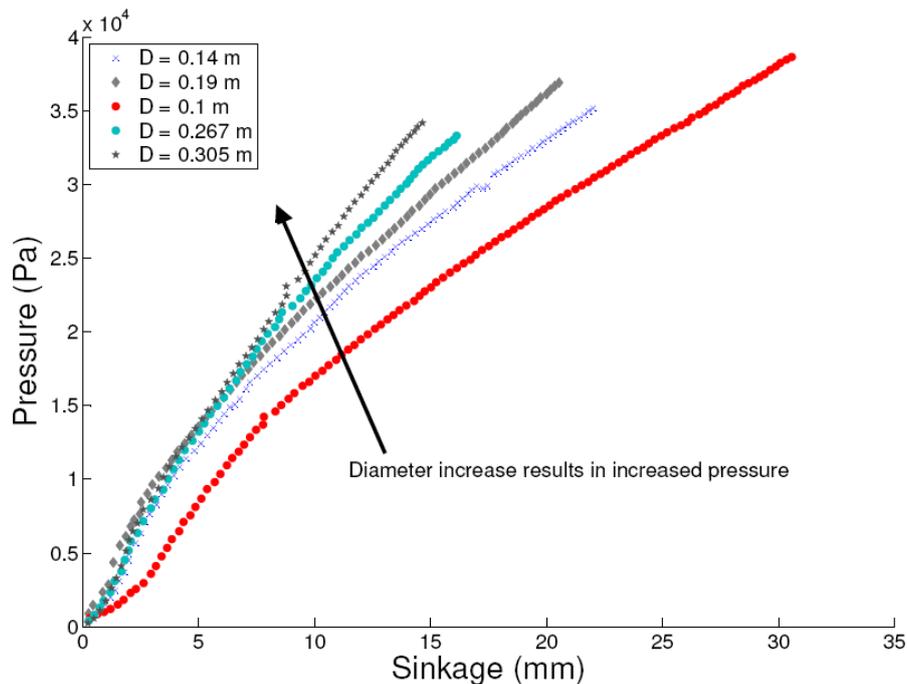


Figure 6: Pressure-sinkage results for dry sand

Figure 6 shows the results for dry sand. The curves exhibit a clear dependence on wheel diameter. As the diameter increases, so does the observed pressure. This implies that **as the wheel diameter increases, the force required to attain a given level of sinkage increases faster than the contact area.**

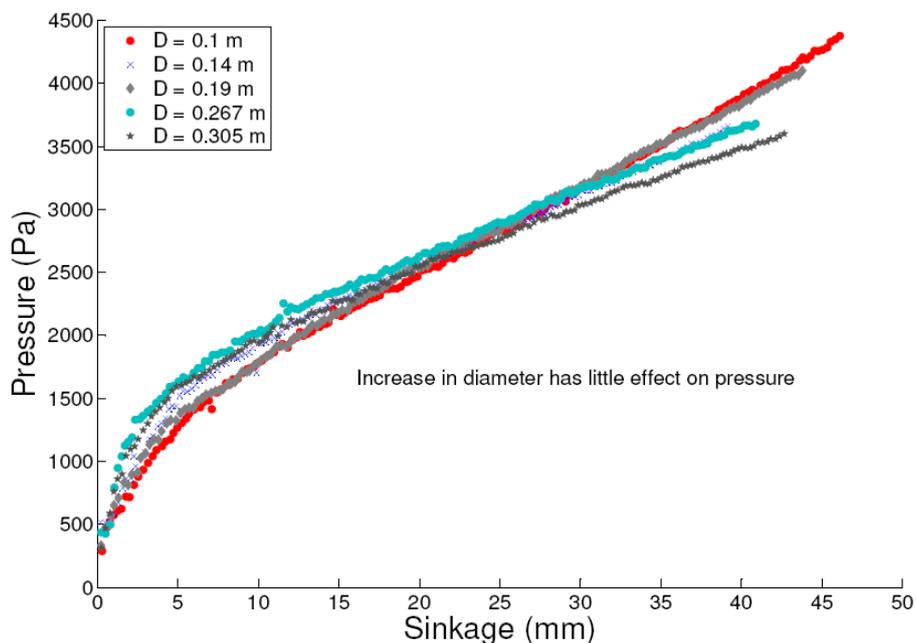


Figure 7: Pressure-sinkage results for calcium silicate

Figure 7 shows the results for calcium silicate, which is a very fine powder and exhibits an almost fluid-like behavior. The force required to achieve total sinkage in this case was around one tenth of that for dry sand. The consequence of this low force is that the overall change in pressure as a function of wheel diameter is very low. This is represented in Figure

7 where the **pressure-sinkage relationship exhibits negligible dependence on wheel diameter.**

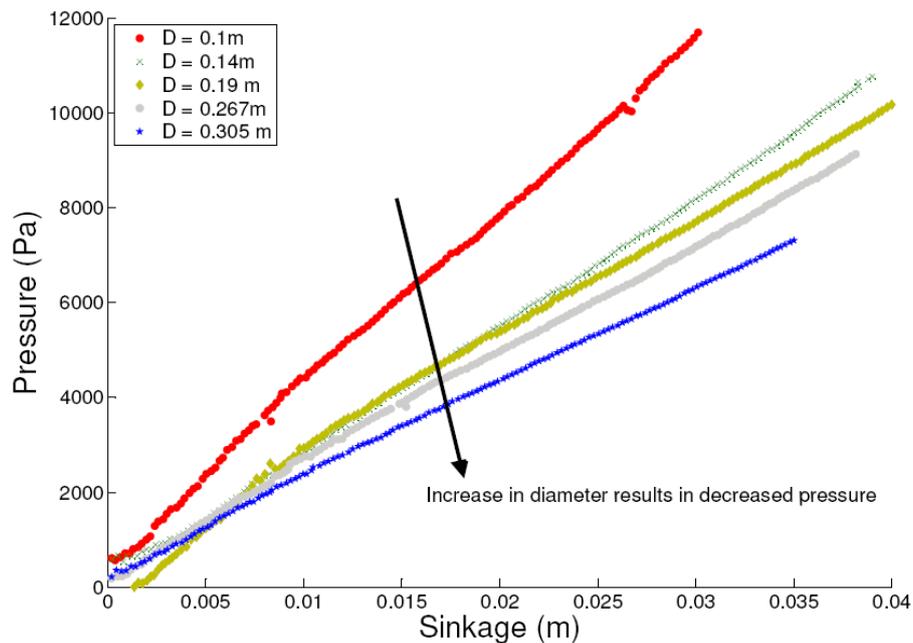


Figure 8: Pressure-sinkage results for moist earth

Figure 8 shows the results for moist earth. The soil moisture content was kept near-constant by performing the tests indoors, over two consecutive days. As with dry sand, the results show a large dependence on wheel diameter. Conversely, as the wheel diameter increases, the pressure decreases. This implies that for moist earth, **as the wheel diameter increases, the contact area increases faster than the force.** Thus, whether the pressure increases or decreases as a function of wheel diameter clearly depends on the properties of the soil being tested.

What is evident in both the dry sand and moist earth results is that a change in wheel diameter has a definite effect. Furthermore, it is clear that this effect manifests itself as a change in the curvature of the pressure-sinkage relationship.

5.1 Improving the Model

Following analysis of the experimental data, a new model is proposed here to account for the dependence on wheel diameter:

$$P = kz^n D^m \quad (9)$$

Here, D is the wheel diameter and m is the diameter exponent. Similarly to the Bernstein-Goriatchkin expression, the proposed model describes a power curve, for which k , n and m are fitting constants. The improvement in the proposed model stems from the inclusion of D^m , which ensures that the curvature of the pressure-sinkage relationship is a function of both sinkage and diameter. The constants k , n and m found for each of the soils tested are shown in Table 6.

Table 6: Proposed model soil properties

	Dry sand	Calcium silicate	Moist earth
Sinkage modulus, k (kN/m ⁿ⁺²)	1604	17	79
Sinkage exponent, n	0.8	0.48	0.88
Diameter exponent, m	0.39	0	-0.49

Note that the parameter, m , can be negative. This occurs when the wheel-soil contact area grows faster than the applied force.

Figure 9 shows an example of the improvement in the pressure-sinkage model when using (9) as opposed to the Bernstein-Goriatchkin equation. Results are shown for the smallest and largest wheel diameters tested.

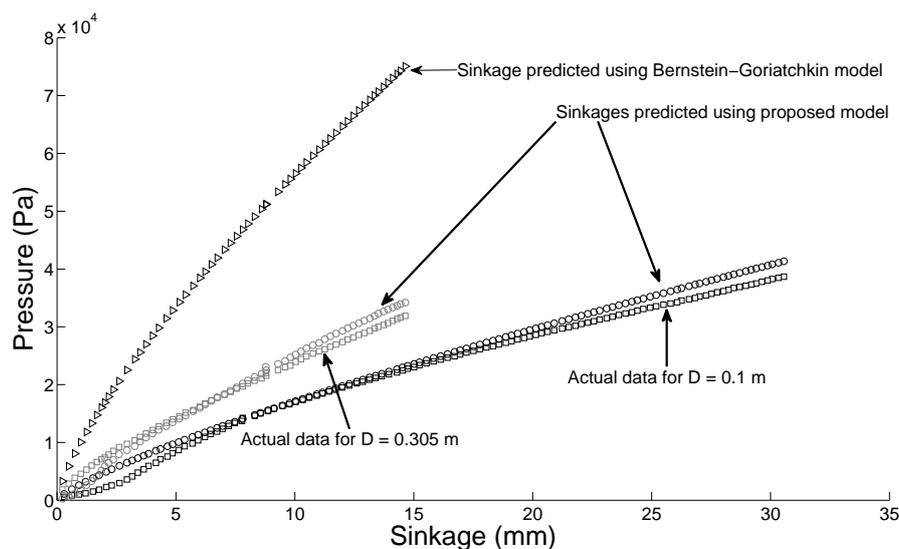


Figure 9: Comparison of Bernstein-Goriatchkin and proposed model on dry sand

Again, we see that the single pressure-sinkage curve offered by the Bernstein-Goriatchkin expression is insufficient when dealing with varying wheel diameters. We also see that the proposed model yields a significantly more accurate model in both cases. This was found to be true for all the tested wheel diameters and soils. Figures 10, 11 and 12 compare the root mean squared error (RMSE) of pressure-sinkage predictions using both the Bernstein-Goriatchkin and proposed models.

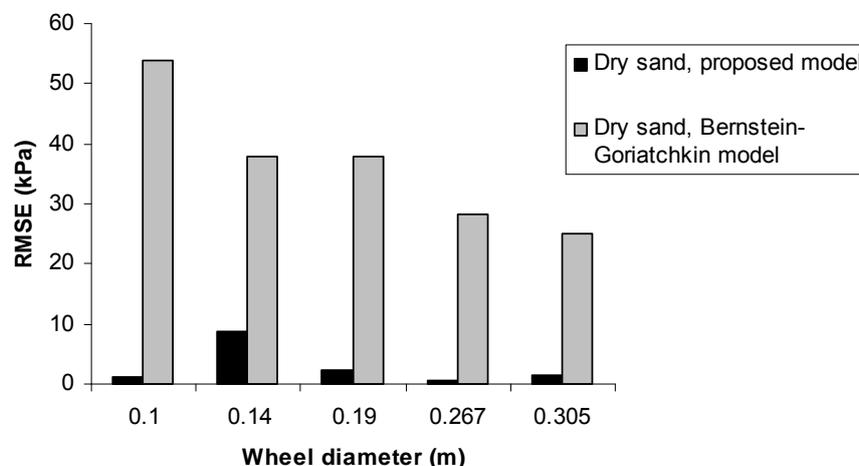


Figure 10: Comparison of RMSE values for dry sand

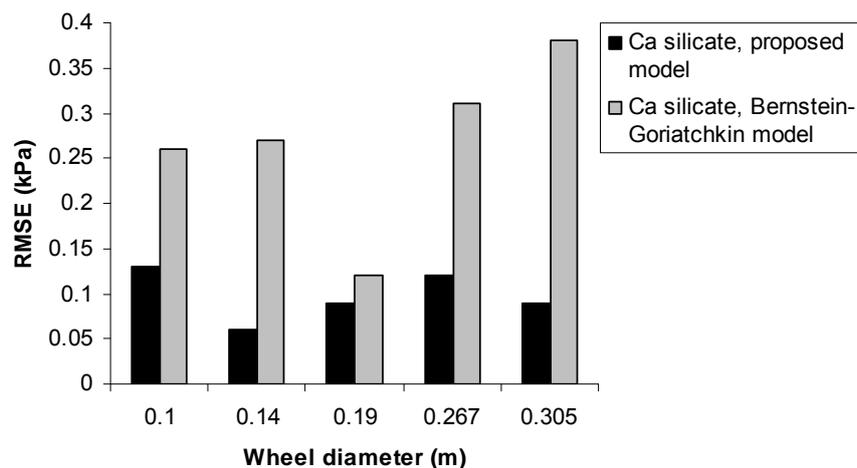


Figure 11: Comparison of RMSE values for calcium silicate

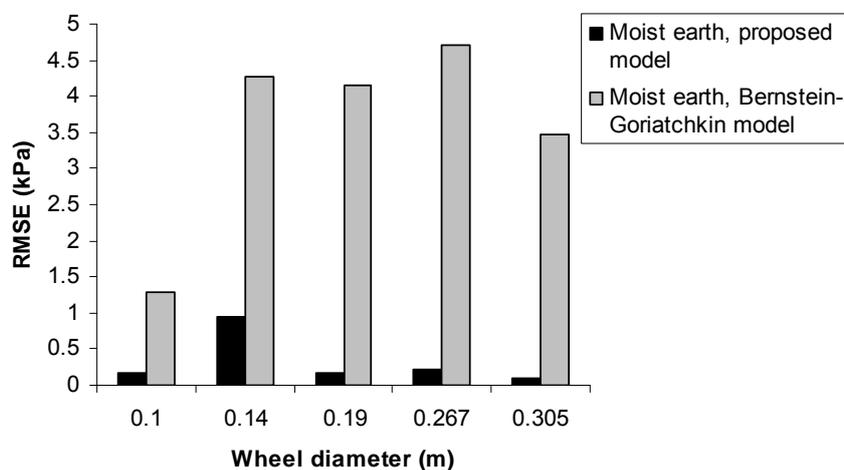


Figure 12: Comparison of RMSE values for moist earth

Clearly, for each of the soils and wheel diameters tested, the proposed model offers a significant improvement in accuracy.

5.2 Updating the Sinkage and Compaction Resistance Models

Substituting the proposed model into the derivation of the sinkage equation as shown in Section 3.1, yields **the new sinkage model**:

$$z_0 = \frac{3W}{b(3-n)kD^{m+0.5}}^{2/2n+1} \quad (10)$$

Solving the pressure-sinkage relationship for small diameter wheels also alters Bekker's compaction resistance equation. Using Figure 2, we can deduce that:

$$R_c = b \int_0^\theta \sigma \sin \theta d\theta \quad (11)$$

which Bekker re-stated as:

$$R_c = b \int_0^{z_0} kz^n dz \quad (12)$$

Notice that this equation relies heavily on the Bernstein-Goriatchkin relationship. As such, we may replace it with our proposed model:

$$R_c = b \int_0^{z_0} kz^n D^m dz \quad (13)$$

and therefore **the new compaction resistance model is:**

$$R_c = bkD^m \frac{z_0^{n+1}}{n+1} \quad (14)$$

5.3 Validation of the Proposed Model

Results obtained from rolling wheel tests performed using the vehicle-terrain testbed are presented. Tests were performed using a rigid wheel of diameter and width 0.17 m and 0.085 m, respectively. These tests were performed to validate the derivation of the sinkage and compaction resistance equations shown in Section 5.2.

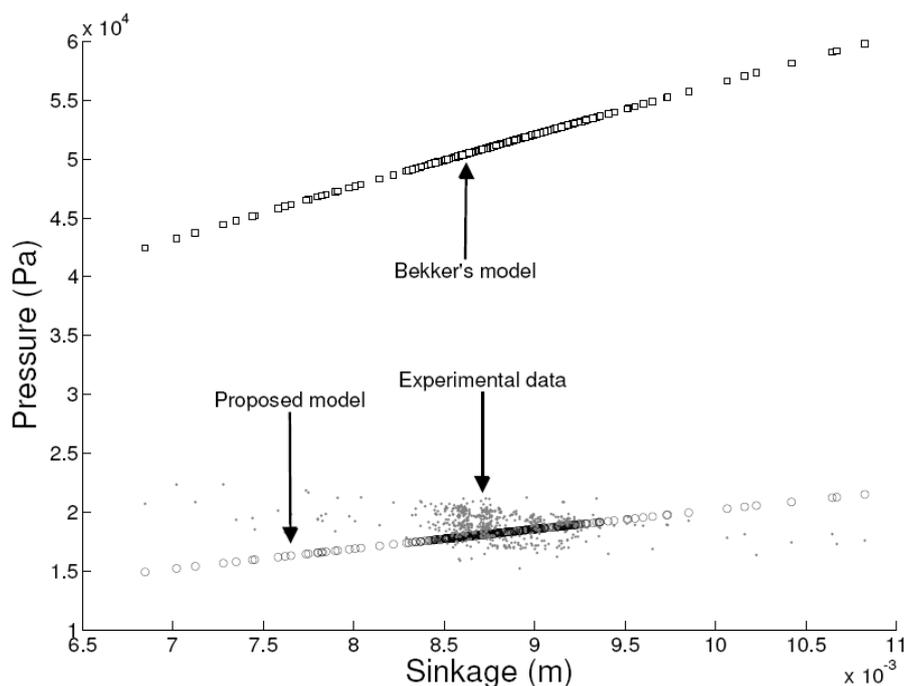


Figure 13: Comparison of experimental sinkage data fit with Bekker and proposed models

In Figure 13, Bekker's equation clearly over-estimates the pressure required to achieve the experimentally found level of sinkage. The proposed model, however, adheres to the pressure-sinkage data with good accuracy.

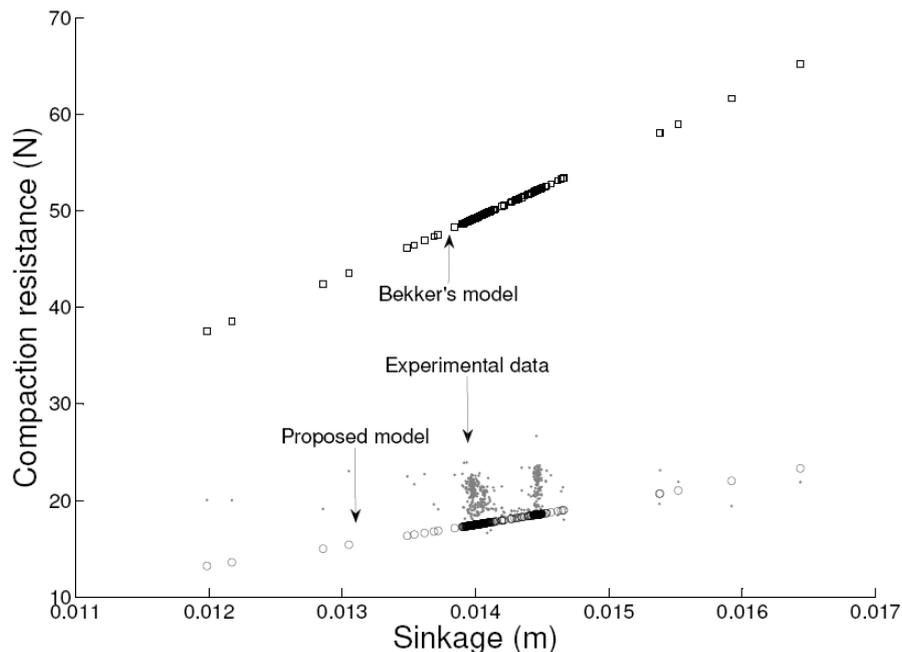


Figure 14: Comparison of experimental sinkage data fit with Bekker and proposed models

For compaction resistance, tests were performed by measuring the motion resistance while towing the wheel. In Figure 14 we see Bekker's equation significantly over-estimates the resistance. Again, we see that using the proposed model reduces the error substantially. Here, a slight under-estimation of resistance by the proposed model is observed. This is attributed to a small level of bulldozing resistance that is not taken into account in this model.

Table 7 summarizes and compares the accuracy of the proposed model vs. Bekker's model for the above tests.

Table 7: Accuracy comparison for Bekker and new models

	Experimental	Bekker	Proposed	Improvement
Sinkage (mm)	8.8	5.2	9.8	3.6x
Compaction resistance (N)	20.8	50.6	17.9	10x

6 Conclusion

In this paper, the limitations of Bekker's model for sinkage and compaction resistance have been discussed. An empirical investigation into the effect of wheel diameter on the pressure-sinkage relationship for small wheels has been detailed. Furthermore, the results of this investigation yielded a new model for both sinkage and compaction resistance. The accuracy of these models has been shown to be significantly higher than that observed using traditional Bekker theory.

Other assumptions, detailed in Section 3 of this paper, are still prevalent in Bekker's sinkage model. In order for this to be improved, these assumptions must be investigated and where necessary, removed. Furthermore, the accuracy of other equations such as thrust and bulldozing resistance, when applied to small wheels is still unknown. These areas must be fully investigated for the application of terramechanics theory to small vehicles to be truly reliable.

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