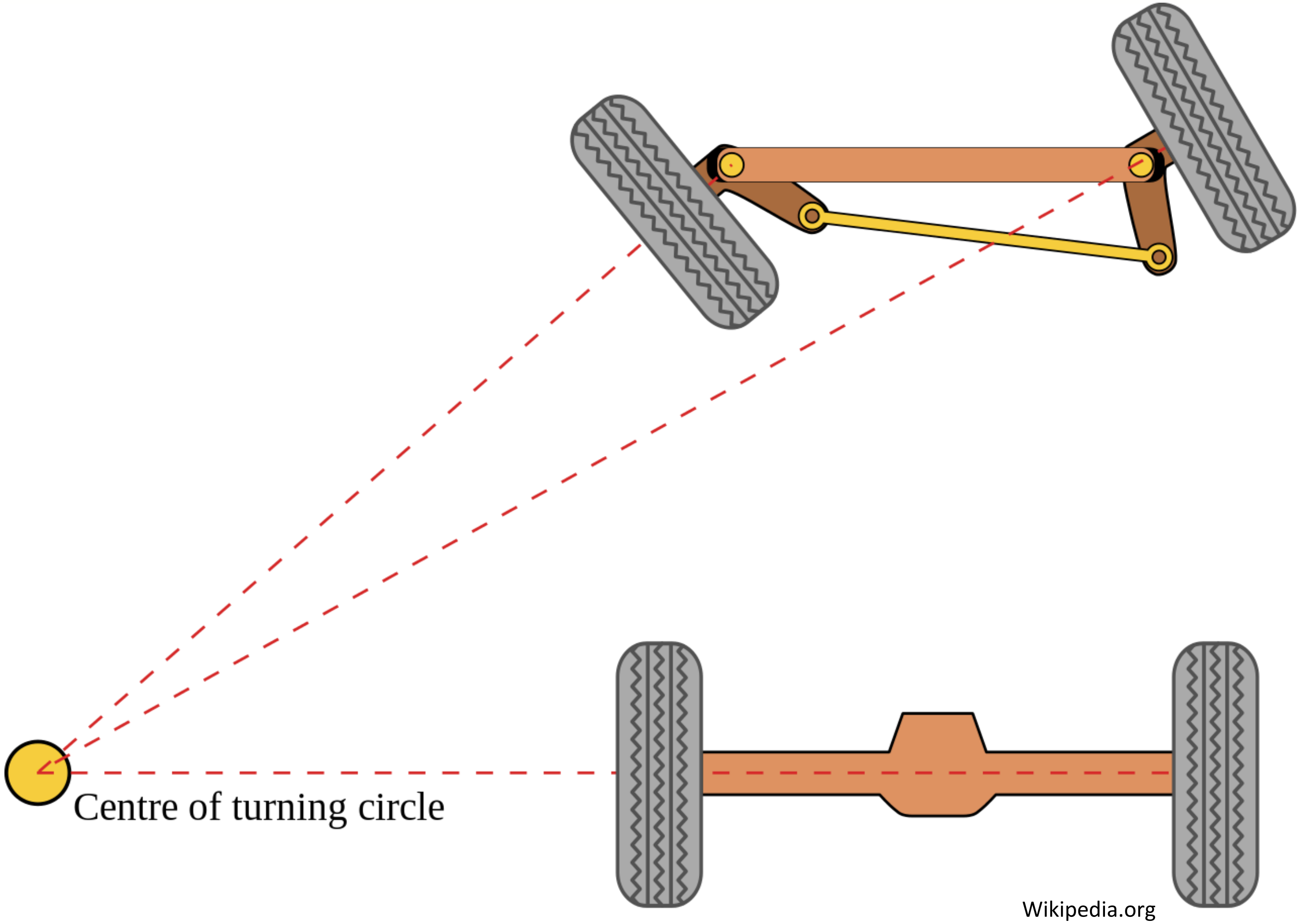
Mobile Robot Kinematics

Prof. Matthew Spenko
MMAE 540: Introduction to Robotics
Illinois Institute of Technology

Mobile Robot Kinematics

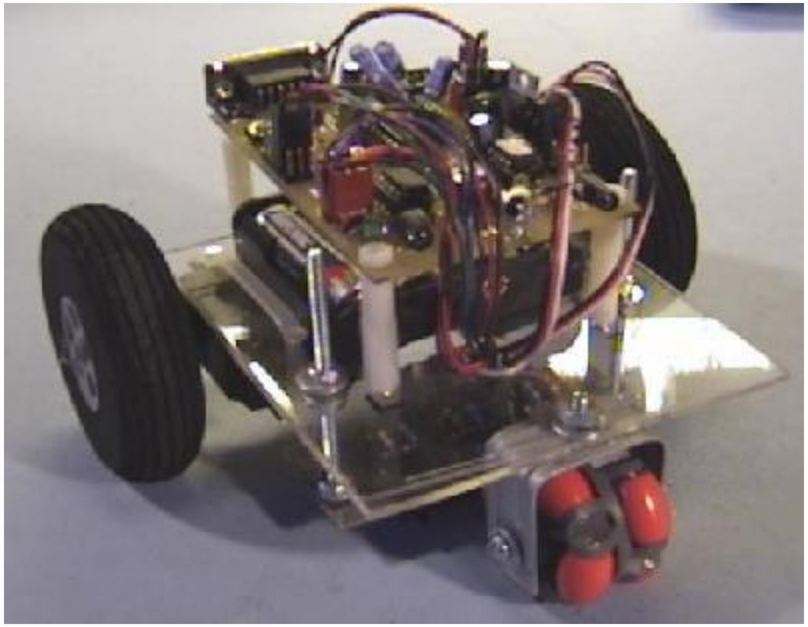
Typical Mobile Robot Types

Ackermann Steered Vehicles



Differential Drive Robots

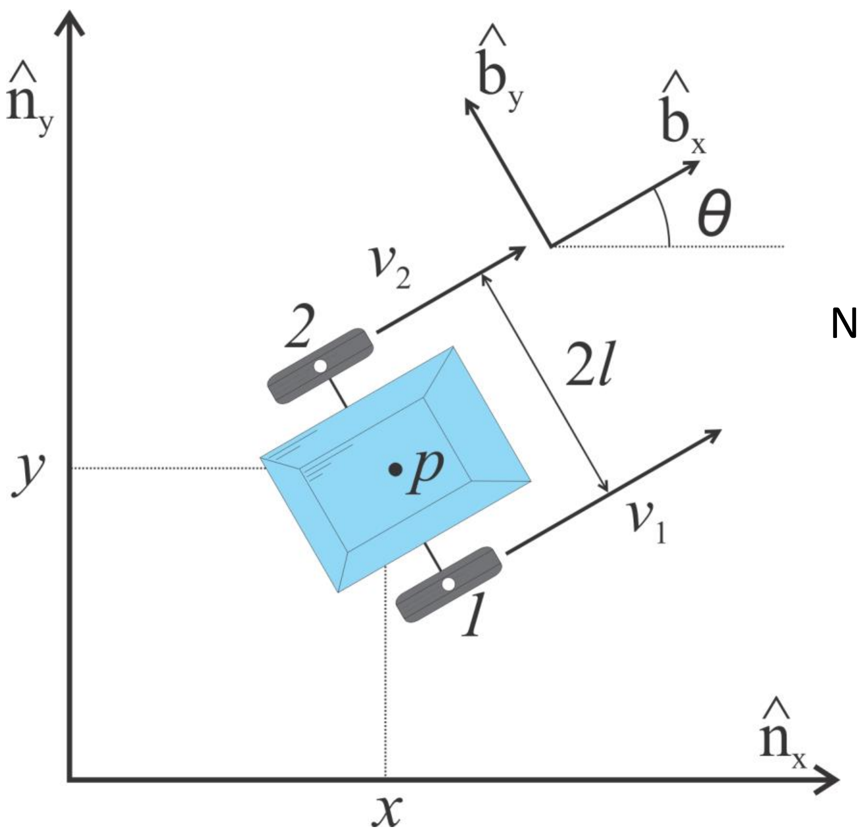
- Popular and common
- 0 turn radius at 0 velocity
- Turn radius is function of velocity



Skid Steered Mobile Robots

- Must induce slip to turn

Example: Differential Drive Kinematics



- Given
  - Wheel radius, R
  - Wheel angular velocity, ω
- Constraints

$$ {}_n\mathbf{v}_1 = \omega_1 R\hat{\mathbf{b}}_x $$
$$ {}_n\mathbf{v}_2 = \omega_2 R\hat{\mathbf{b}}_x $$

About the **b**$_x$ axis

N-observed velocity

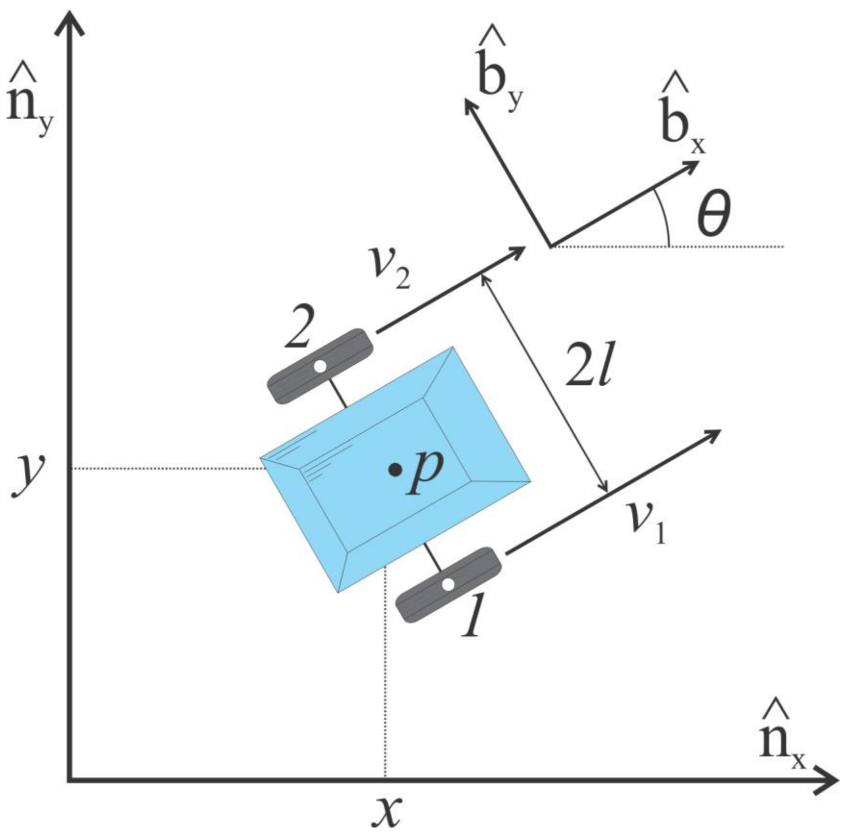
of point 2

Angular velocity Of point 2 * R

- Find the forward kinematic model:

$$ \dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} $$

Example: Differential Drive Kinematics



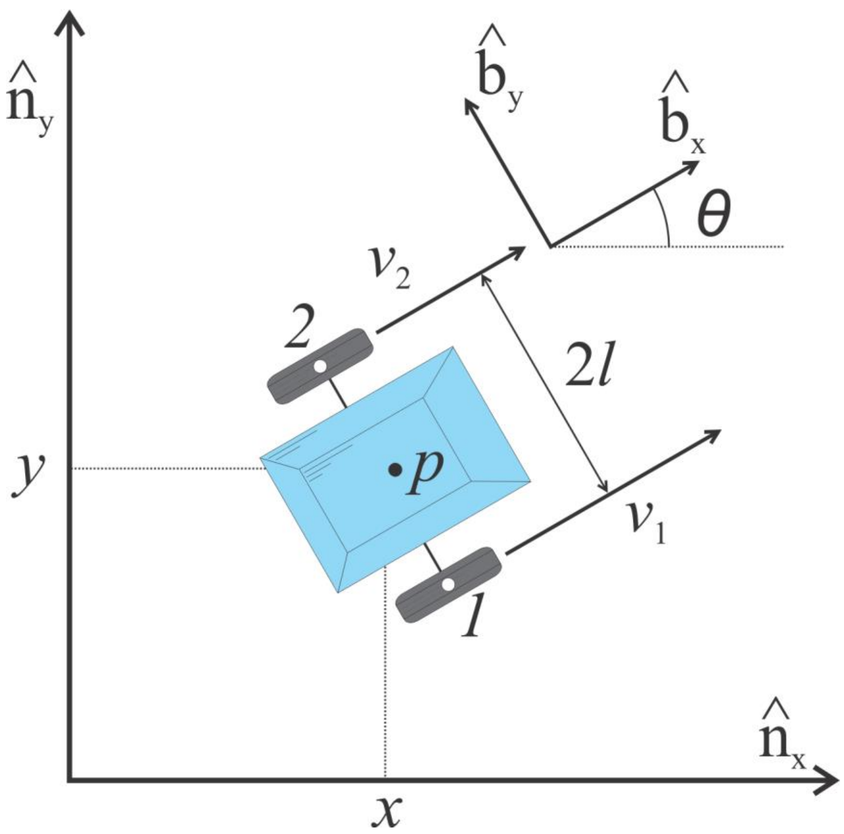
- Solve for heading angle velocity

$$ {}_n\mathbf{v}_1 = {}_n\mathbf{v}_2 + {}_n\omega_B \times \mathbf{r}_{1/2} $$

$$ \omega_1 R\hat{\mathbf{b}}_x = \omega_2 R\hat{\mathbf{b}}_x + \det \begin{vmatrix} \hat{\mathbf{b}}_x & \hat{\mathbf{b}}_y & \hat{\mathbf{b}}_z \\ 0 & 0 & \dot{\theta} \\ 0 & -2l & 0 \end{vmatrix} $$

$$ \dot{\theta} = \frac{R\left(\omega_1 - \omega_2\right)}{2l} $$

Example: Differential Drive Kinematics



- Solve for velocity

$$
\begin{aligned}
{}_n\mathbf{v}_p &= {}_n\mathbf{v}_1 + {}_n\omega_B \times \mathbf{r}_{p/1} \\
&= \omega_1 R \hat{\mathbf{b}}_x + \det \begin{vmatrix} \hat{\mathbf{b}}_x & \hat{\mathbf{b}}_y & \hat{\mathbf{b}}_z \\ 0 & 0 & \frac{R(\omega_1 - \omega_2)}{2l} \\ 0 & l & 0 \end{vmatrix} \\
&= \frac{R\left(\omega_1 + \omega_2\right)}{2} \hat{\mathbf{b}}_x
\end{aligned}
$$

Example: Differential Drive Kinematics

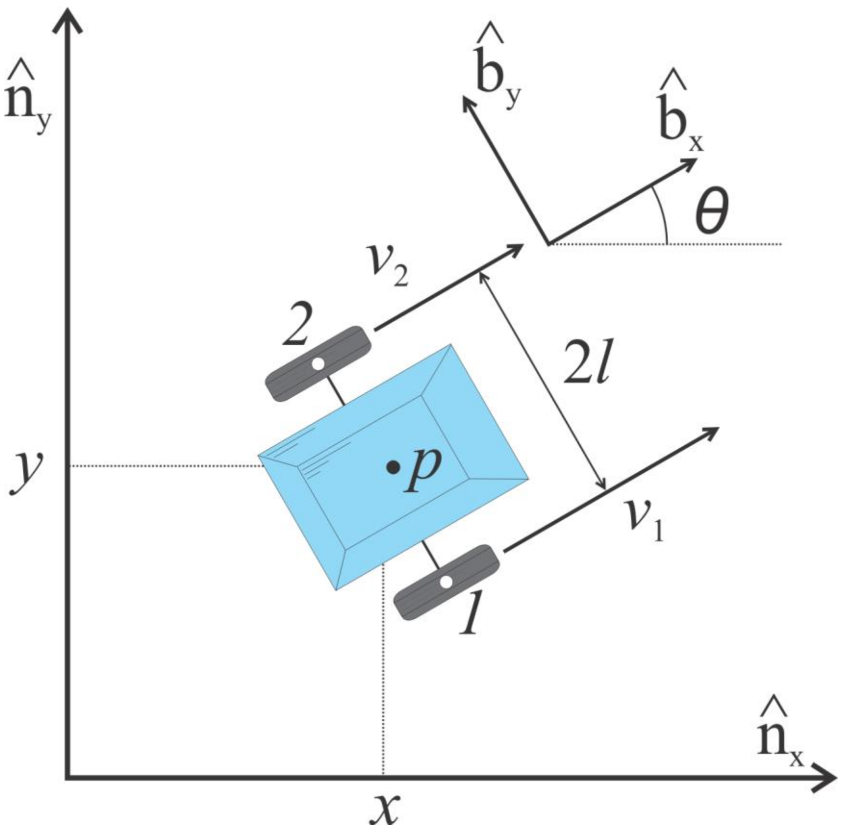
$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

$${}_n\mathbf{v}_p = \frac{R\left(\omega_1 + \omega_2\right)}{2}\hat{\mathbf{b}}_x$$

$$\dot{\theta} = \frac{R\left(\omega_1 - \omega_2\right)}{2l}$$

Example: Differential Drive Kinematics

- Transform ${}_n\mathbf{v}_p$ into the inertial frame to obtain $\dot{x}, \dot{y}$



${}^n\mathbf{R}^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$	$\hat{\mathbf{b}}_z$
$\hat{\mathbf{n}}_x$	$\cos\theta$	$-\sin\theta$	0
$\hat{\mathbf{n}}_y$	$\sin\theta$	$\cos\theta$	0
$\hat{\mathbf{n}}_z$	0	0	1

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = {}^n\mathbf{R}^b \begin{bmatrix} \frac{R(\omega_1+\omega_2)}{2} \\ 0 \\ \frac{R(\omega_1-\omega_2)}{2l} \end{bmatrix}$$

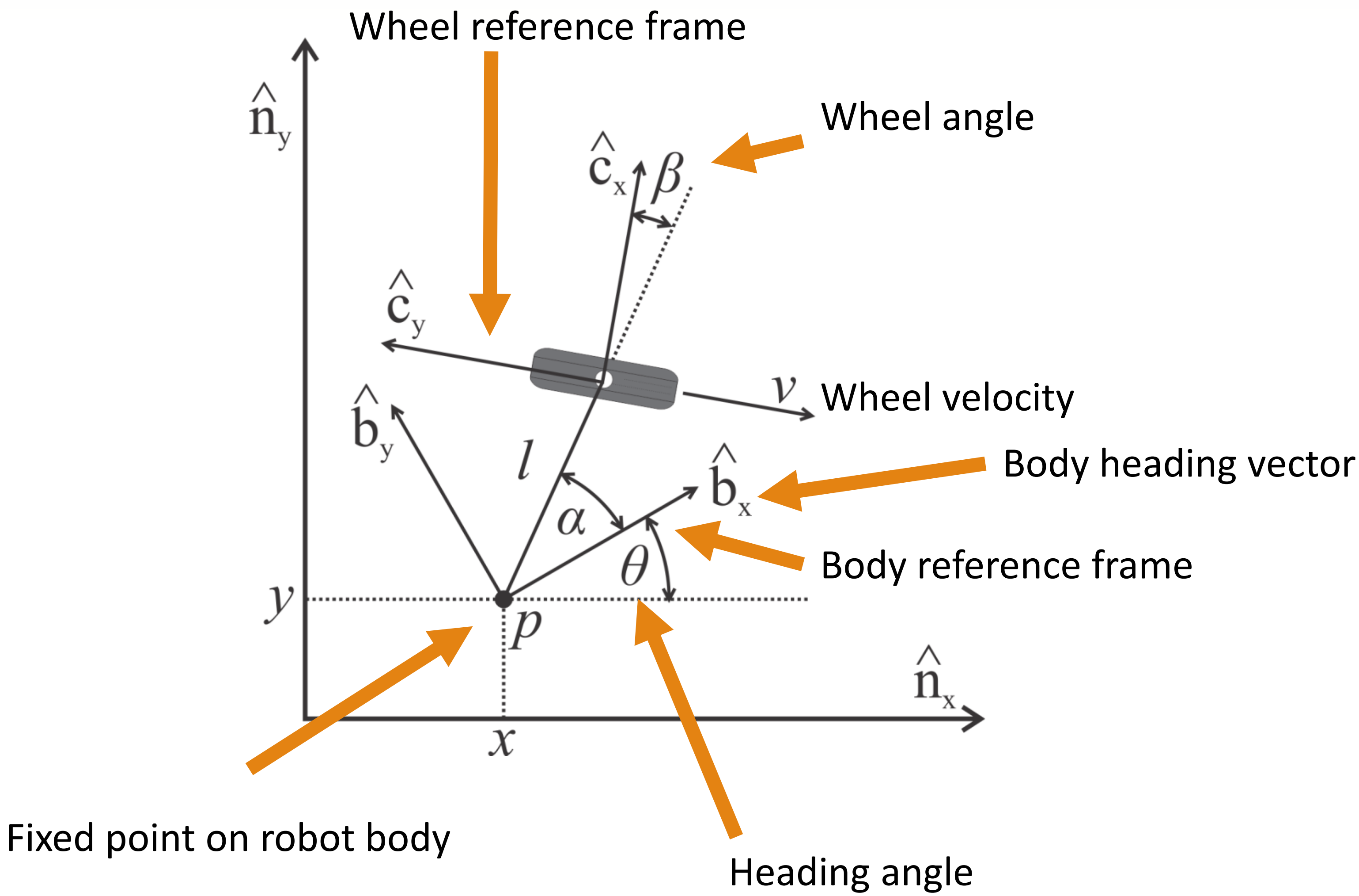
Mini Quiz

- Working alone, answer the following:
  - Schematically draw an Ackermann-Steered vehicle, differential drive vehicle, and skid-steered vehicle
  - What is the difference between a differential drive and skid steered vehicle?
  - Describe all of the terms in:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = {}^{n}\mathbf{R}^{b} \begin{bmatrix} \frac{R(\omega_1+\omega_2)}{2} \\ 0 \\ \frac{R(\omega_1-\omega_2)}{2l} \end{bmatrix}$$

- After you are done, discuss your results with your neighbor

Generalized Fixed Wheel Kinematics



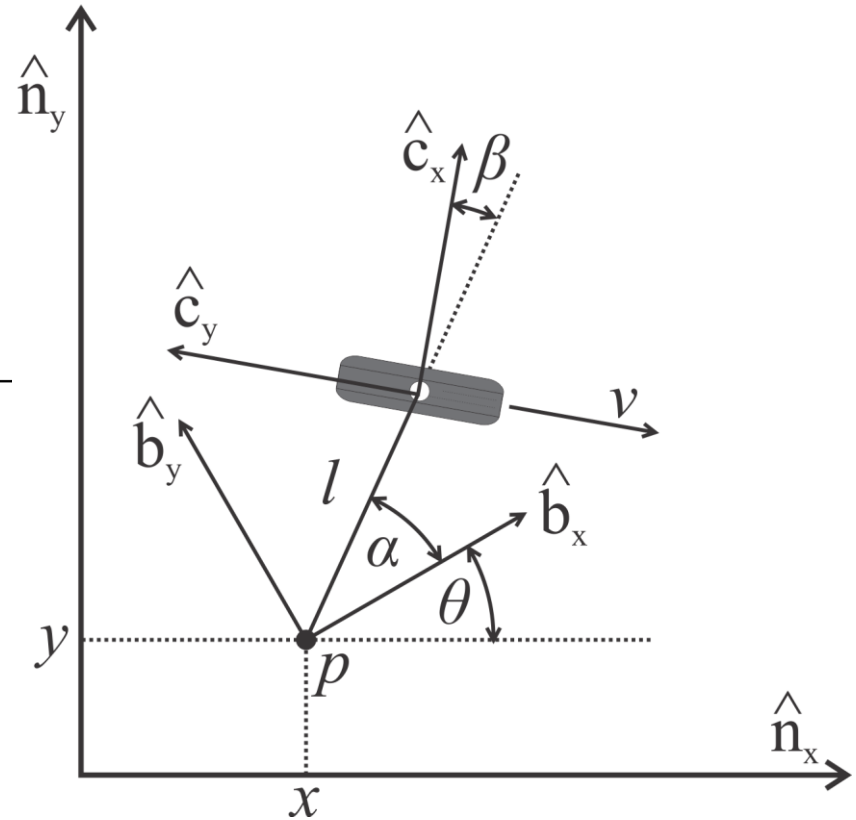
Generalized Fixed Wheel Kinematics

- Define the rotation matrices

${}^b\mathbf{R}^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$
$\hat{\mathbf{b}}_x$	$\cos\theta$	$\sin\theta$
$\hat{\mathbf{b}}_y$	$-\sin\theta$	$\cos\theta$

${}^c\mathbf{R}^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$
$\hat{\mathbf{c}}_x$	$\cos(\alpha+\beta)$	$\sin(\alpha+\beta)$
$\hat{\mathbf{c}}_y$	$-\sin(\alpha+\beta)$	$\cos(\alpha+\beta)$

${}^c\mathbf{R}^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$
$\hat{\mathbf{c}}_x$	$c_{\alpha\beta\theta}$	$s_{\alpha\beta\theta}$
$\hat{\mathbf{c}}_y$	$-s_{\alpha\beta\theta}$	$c_{\alpha\beta\theta}$



Generalized Fixed Wheel Kinematics

- Find the velocity of the wheel

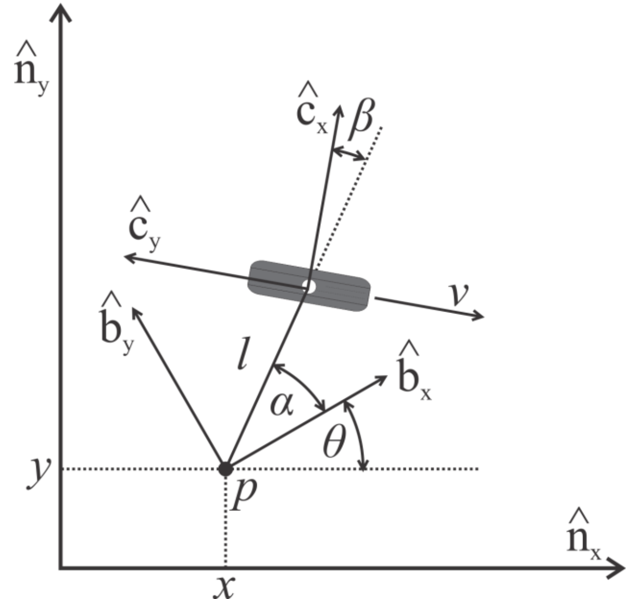
$$ {}_{n}\mathbf{v}_{wheel} = {}_{n}\mathbf{v}_{p} + \frac{{}^{B}d\mathbf{r}_{wheel/p}}{dt} + {}_{n}\boldsymbol{\omega}_{b} \times \mathbf{r}_{wheel/p} $$

$$ -\dot{\phi} r \hat{\mathbf{c}}_y = \dot{x}\hat{\mathbf{n}}_x + \dot{y}\hat{\mathbf{n}}_y + \frac{{}^{B}d\left(l\cos\alpha\hat{\mathbf{b}}_x + l\sin\alpha\hat{\mathbf{b}}_y\right)}{dt} + {}_{N}\omega_{B} \times \left(l\cos\alpha\hat{\mathbf{b}}_x + l\sin\alpha\hat{\mathbf{b}}_y\right) $$

$$ -\dot{\phi}\hat{\mathbf{c}}_y = \dot{x}\hat{\mathbf{n}}_x + \dot{y}\hat{\mathbf{n}}_y + \det\begin{vmatrix} \hat{\mathbf{b}}_x & \hat{\mathbf{b}}_y & \hat{\mathbf{b}}_z \\ 0 & 0 & \dot{\theta} \\ l\cos\alpha & l\sin\alpha & 0 \end{vmatrix} $$

$$ -\dot{\phi} r \hat{\mathbf{c}}_y = \dot{x}\hat{\mathbf{n}}_x + \dot{y}\hat{\mathbf{n}}_y - l\dot{\theta}\sin\alpha\hat{\mathbf{b}}_x + l\dot{\theta}\cos\alpha\hat{\mathbf{b}}_y $$

Constraint equation



Generalized Fixed Wheel Kinematics Continued

- Need this in the C-basis

$$-\dot{\phi} r \hat{\mathbf{c}}_y = \dot{x}\hat{\mathbf{n}}_x + \dot{y}\hat{\mathbf{n}}_y - l\dot{\theta}\sin\alpha\hat{\mathbf{b}}_x + l\dot{\theta}\cos\alpha\hat{\mathbf{b}}_y$$

${}^c\mathbf{R}^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$
$\hat{\mathbf{c}}_x$	$\cos(\alpha+\beta)$	$\sin(\alpha+\beta)$
$\hat{\mathbf{c}}_y$	$-\sin(\alpha+\beta)$	$\cos(\alpha+\beta)$

${}^c\mathbf{R}^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$
$\hat{\mathbf{c}}_x$	$c_{\alpha\beta\theta}$	$s_{\alpha\beta\theta}$
$\hat{\mathbf{c}}_y$	$-s_{\alpha\beta\theta}$	$c_{\alpha\beta\theta}$

$$-\dot{\phi} r \hat{c}_y = \dot{x}c_{\alpha\beta\theta}\hat{\mathbf{c}}_x - \dot{x}s_{\alpha\beta\theta}\hat{\mathbf{c}}_y + \dot{y}s_{\alpha\beta\theta}\hat{\mathbf{c}}_x$$
$$+\dot{y}c_{\alpha\beta\theta}\hat{\mathbf{c}}_y - l\dot{\theta}s_\alpha c_{\alpha\beta}\hat{\mathbf{c}}_x + l\dot{\theta}s_\alpha s_{\alpha\beta}\hat{\mathbf{c}}_y + l\dot{\theta}c_\alpha s_{\alpha\beta}\hat{\mathbf{c}}_x + l\dot{\theta}c_\alpha c_{\alpha\beta}\hat{\mathbf{c}}_y$$

- The c_y component gives the pure rolling constraint

$$-\dot{\phi} r = -\dot{x}s_{\alpha\beta\theta} + \dot{y}c_{\alpha\beta\theta} + l\dot{\theta}\left(s_\alpha s_{\alpha\beta} + c_\alpha c_{\alpha\beta}\right)$$
$$-\dot{\phi} r = -\dot{x}s_{\alpha\beta\theta} + \dot{y}c_{\alpha\beta\theta} + l\dot{\theta}\left(c_\beta\right)$$

Generalized Fixed Wheel Kinematics Continued

- Rearrange:

$$-\dot{\phi} r = -\dot{x} s_{\alpha\beta\theta} + \dot{y} c_{\alpha\beta\theta} + l\dot{\theta}\left(c_{\beta}\right)$$

$$\dot{\phi} r = \left[\begin{array}{ccc} s_{\alpha\beta} & -c_{\alpha\beta} & -lc_{\beta} \end{array}\right] {}^{b}R^{n} \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array}\right]$$

- Verify:

$$\dot{\phi} r = \left[\begin{array}{ccc} \underbrace{s_{\alpha\beta} c_{\theta} + c_{\alpha\beta} s_{\theta}} & s_{\alpha\beta} s_{\theta} - c_{\alpha\beta} c_{\theta} & -lc_{\beta} \end{array}\right] \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array}\right]$$

$$s_{\alpha\beta\theta}$$

- Why? Separates wheels (each with its own α and β) from body reference frame

Generalized Fixed Wheel Kinematics Continued

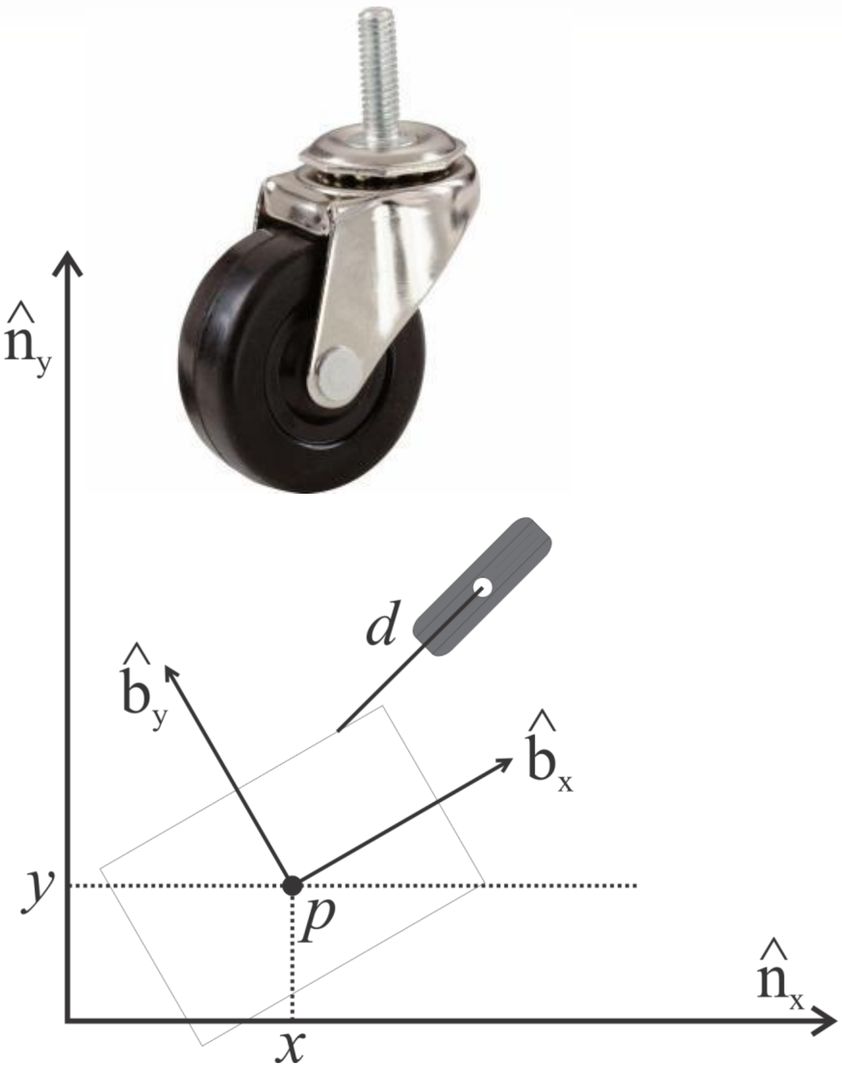
- Now just the c_x component

$$-\dot{\phi} r \hat{c}_y = \dot{x} c_{\alpha\beta\theta} \hat{\mathbf{c}}_x - \dot{x} s_{\alpha\beta\theta} \hat{\mathbf{c}}_y + \dot{y} s_{\alpha\beta\theta} \hat{\mathbf{c}}_x$$
$$+\dot{y} s_{\alpha\beta\theta} \hat{\mathbf{c}}_y - l\dot{\theta} s_\alpha c_{\alpha\beta} \hat{\mathbf{c}}_x + l\dot{\theta} s_\alpha s_{\alpha\beta} \hat{\mathbf{c}}_y + l\dot{\theta} c_\alpha s_{\alpha\beta} \hat{\mathbf{c}}_x + l\dot{\theta} c_\alpha c_{\alpha\beta} \hat{\mathbf{c}}_y$$

$$0 = \begin{bmatrix} c_{\alpha\beta} & s_{\alpha\beta} & l s_\beta \end{bmatrix} {}^b R^n \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

- Gives the no sideslip constraint
- Steerable wheel = same constraints, but now $\beta = \beta(t)$

Caster wheel



- Rolling constraint is the same
- Sliding constraint

$$0 = \left[\begin{array}{ccc} c_{\alpha\beta} & s_{\alpha\beta} & d + l s_{\beta} \end{array} \right] {}^{b}R^{n} \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array} \right]$$

- An omnidirectional system because for all

$$\dot{\xi} = \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array} \right]$$

there exists some value of $\dot{\beta}$ and $\dot{\phi}$ such that the constraints are met

Mobile Robot Kinematics Mini Quiz

- Describe what a rolling and no side-slip constraint physically mean
- Describe the terms in the following constraint equation:

$$0 = \left[\begin{array}{ccc} c_{\alpha\beta} & s_{\alpha\beta} & l s_{\beta} \end{array} \right] {}^{b}R^{n} \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array} \right]$$

- What type of constraint is this (i.e. fixed, steerable, rolling, no-sideslip, caster)?

Mobile Robot Chassis Constraints

- Now have all of the constraints
- Can then compute constraints of entire robot chassis
- Combine all kinematic constraints from each wheel
- No need to consider caster wheels, only standard and steerable wheels

Mobile Robot Chassis constraints

- Suppose the robot has N standard wheels
- N composed of N_f fixed wheels and N_s steerable wheels
- Let $\beta_s(t)$ be the variable steering angles of the steerable wheels
- Let β_f be the fixed steering angel of the fixed wheels
- Let $\phi_f(t)$ be the rotational positon of the steerable wheels
- Let $\phi_s(t)$ be the rotational position of the fixed wheels
- Let $$\phi(t) = \begin{bmatrix} \phi_f(t) \\ \phi_s(t) \end{bmatrix}$$

Rolling Constraint

- The rolling constraints can now be written as:

$$\mathbf{J}_1(\beta_s)\mathbf{R}\dot{\xi}_I - \mathbf{J}_2\dot{\phi} = \mathbf{0}$$

Matrix represents all projections for all wheels to their motions along the individual wheel planes

Constant Diagonal
NxN matrix whose entrants are:

$$\begin{bmatrix} r_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_n \end{bmatrix}$$

$$\mathbf{J}_1(\beta_s) = \begin{bmatrix} \mathbf{J}_{1f} \\ \mathbf{J}_{1s}(\beta_s) \end{bmatrix}$$

N_f x 3

N_s x 3

all wheels must spin about the horizontal axis an appropriate amount so that rolling occurs

No Side Slip Constraint

- Can use the same technique to get the no side slip constraint:

$$\mathbf{C}_1\left(\boldsymbol{\beta}_s\right)\mathbf{R}\left(\theta\right)\dot{\xi}_I = 0$$

- Where:

$$\mathbf{C}_1\left(\beta_s\right) = \left[\begin{array}{c} \mathbf{C}_{1f} \\ \mathbf{C}_{1s}\left(\beta_s\right) \end{array}\right]$$

- Combine to yield full set of kinematic constraints:

$$\left[\begin{array}{c} \mathbf{J}_1\left(\beta_s\right) \\ \mathbf{C}_1\left(\beta_s\right) \end{array}\right]\mathbf{R}\left(\theta\right)\dot{\xi}_I = \left[\begin{array}{c} \mathbf{J}_2\left(\phi\right) \\ \mathbf{0} \end{array}\right]$$

Example: Differential Drive Robot

- Two fixed wheels
- Fixed wheel constraint equations:

$$0 = \begin{bmatrix} s_{\alpha\beta} & -c_{\alpha\beta} & -lc_{\beta} \end{bmatrix} {}^{b}R^{n} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} - \dot{\phi} r$$

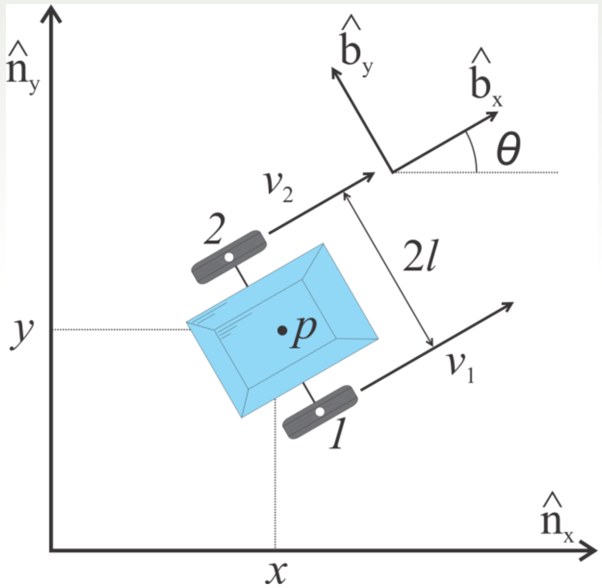
$$0 = \begin{bmatrix} c_{\alpha\beta} & s_{\alpha\beta} & ls_{\beta} \end{bmatrix} {}^{b}R^{n} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

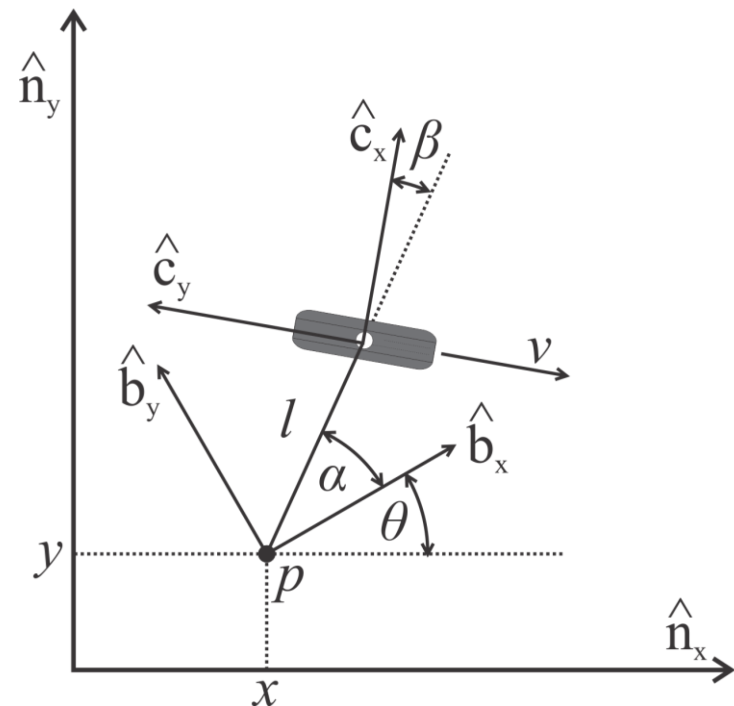
- Wheel 1

$$\alpha_1 = -\frac{\pi}{2} \quad \beta_1 = \pi$$

- Wheel 2

$$\alpha_2 = \frac{\pi}{2} \quad \beta_2 = 0$$





$$0 = \left[\begin{array}{ccc} s_{\alpha\beta} & -c_{\alpha\beta} & -lc_\beta \end{array}\right] {}^bR^n \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array}\right] - \dot{\phi} r$$

$$\alpha_2 = \frac{\pi}{2} \quad \beta_2 = 0$$

$$\alpha_1 = -\frac{\pi}{2} \quad \beta_1 = \pi$$

$$0 = \left[\begin{array}{ccc} c_{\alpha\beta} & s_{\alpha\beta} & ls_\beta \end{array}\right] {}^bR^n \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array}\right]$$

$$\begin{array}{l} \mathbf{J} \\ \mathbf{C} \end{array} \quad \left[\begin{array}{ccc} \sin\left(-\frac{\pi}{2} + \pi\right) & -\cos\left(-\frac{\pi}{2} + \pi\right) & -l\cos\pi \\ \sin\left(\frac{\pi}{2} + 0\right) & -\cos\left(\frac{\pi}{2} + 0\right) & -l\cos 0 \\ \cos\left(\frac{\pi}{2} + 0\right) & \sin\left(\frac{\pi}{2} + 0\right) & l\sin 0 \end{array}\right] {}^b\mathbf{R}^N \dot{\xi}_I = \left[\begin{array}{c} r\dot{\phi}_1 \\ r\dot{\phi}_2 \\ 0 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{array}\right] {}^b\mathbf{R}^N \dot{\xi}_I = \left[\begin{array}{c} r\dot{\phi}_1 \\ r\dot{\phi}_2 \\ 0 \end{array}\right]$$

Only including 1 no-side slip constraint because 2nd constraint is identical

Mobile Robot Kinematics

Kinematic Mobility

Degree of Mobility

- Look at the constraint equation for no side slip
- Fixed wheel: $\mathbf{C}_{1f}\,{}^{b}\mathbf{R}^{n}\dot{\xi}_{I} = 0$
- Steerable wheel: $\mathbf{C}_{1s}\left(\beta_s\right){}^{b}\mathbf{R}^{n}\dot{\xi}_{I} = 0$

$${}^{b}\mathbf{R}^{n}\dot{\xi}_{I} = 0$$

- For either constraint to be satisfied, the “motion vector” must belong to the null space of the “projection matrix”

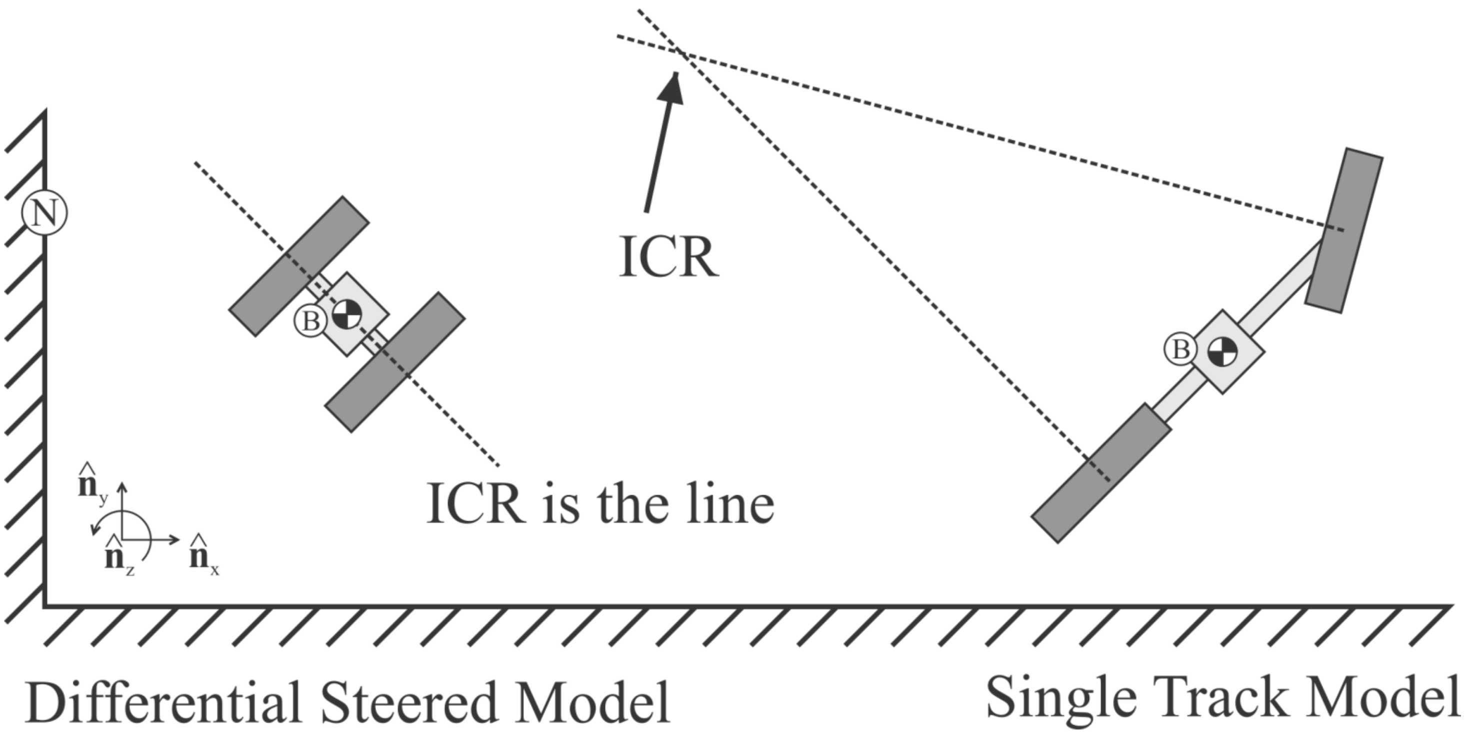
$$\mathbf{C}_{1s}\left(\beta_s\right)$$

- The null space of $\mathbf{C}_{1s}\left(\beta_s\right)$ is the space N such that for any vector:

$$n \in N : C_1\left(\beta_s\right) n = 0$$

Geometric Meaning

- For the constraints to be satisfied the robot must have an instantaneous center of rotation
- The ICR is the point (or set of points) of which the body-observed and Newtonian-observed velocity is the same



Implications of Instantaneous Center of Rotation

- Robot mobility is a function constraint #, not wheel #
  - Ackermann steering = 4 wheels, 3 constraints
  - Single track (bicycle) = 2 wheels, 2 constraints
  - Differential drive = 2 wheels, but 1 constraint
- Robots chassis' kinematics = function of the set of independent constraints from all **standard** wheels
- Related to constraint matrix rank = # of individual constraints
- Higher rank = lower mobility
- Example: unicycle

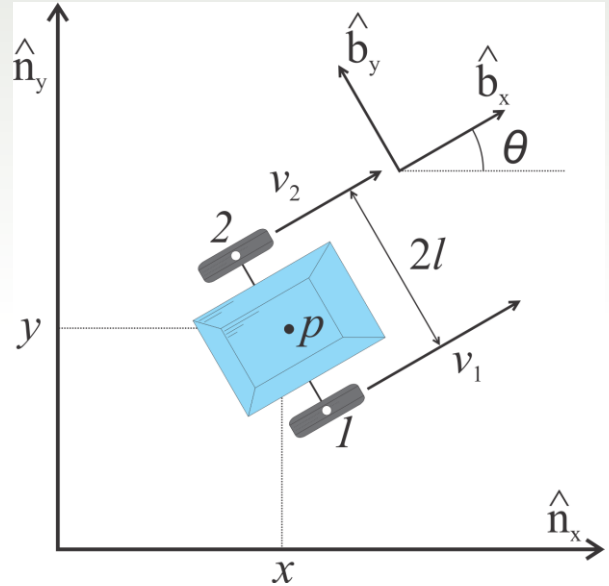
$$\mathbf{C}_1(\beta_s) = \begin{bmatrix} \mathbf{C}_{1f} \\ \mathbf{C}_{1s} \end{bmatrix}$$

← empty

$$\mathbf{C}_1(\beta_s) = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \end{bmatrix}$$

Rank = 1

Example 2



- Differential Drive

$$\mathbf{C}_1(\beta_s) = \begin{bmatrix} c_{\alpha_1} & s_{\alpha_1} & 0 \\ c_{\alpha_1+\pi} & s_{\alpha_1+\pi} & 0 \end{bmatrix}$$

$$\alpha_1 = -\frac{\pi}{2} \quad \beta_1 = \pi \quad \alpha_1 = \frac{\pi}{2} \quad \beta_1 = 0$$

$$\mathbf{C}_1(\beta_s) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- 2 constraints, but rank = 1
- In general:
  - If rank of $\mathbf{C}_{1f} > 1$ then vehicle can only follow a curved or straight path
  - Not very interesting
  - Most systems have fixed and steerable wheels so

$$0 < rank\left[\mathbf{C}_1(\beta_s)\right] \leq 3$$

Corresponds to a completely degenerate vehicle that can't move

Degree of Mobility

- Definition:

$$\underbrace{\delta_m \equiv 3 - rank\left[\mathbf{C}_1\left(\beta_s\right)\right]}$$

Dimension of the null space

- Mobility must lie between 0 and 3
- Examples
  - Differential drive, degree of mobility = 2 (1 constraint)
  - Bicycle, degree of mobility = 1 (2 constraints)

Degree of Steerability and Maneuverability

- Degree of Steerability:

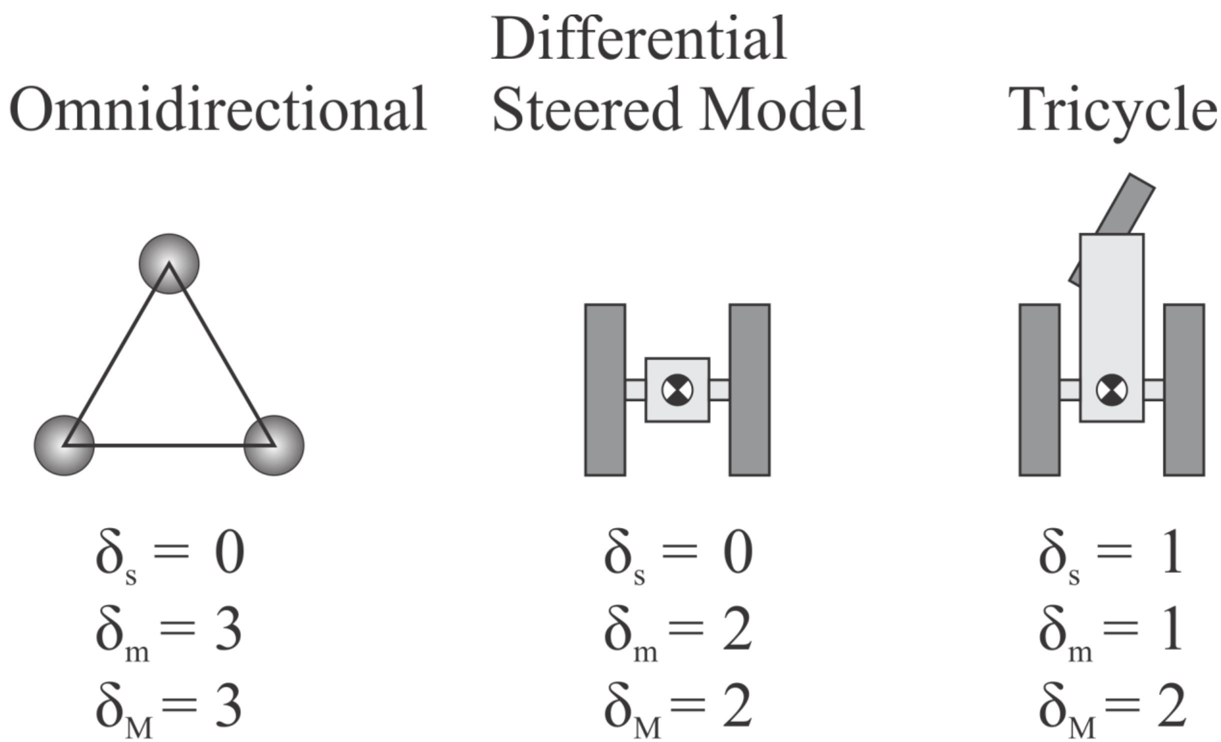
$$\delta_s \equiv rank\left[\mathbf{C}_{1s}\left(\beta_s\right)\right]$$

- An increase in the degree of steerability = more degrees of steering freedom such that a standard wheel can have both low mobility and high steerability
- Maneuverability = degrees of freedom:

$$\delta_M = \delta_m + \delta_s$$

- Maneuverability:
  - 3 = holonomic
  - 2 or less = non-holonomic

Examples


Omnidirectional
Differential Steered Model
Tricycle
$\delta_s = 0$
$\delta_m = 3$
$\delta_M = 3$
$\delta_s = 0$
$\delta_m = 2$
$\delta_M = 2$
$\delta_s = 1$
$\delta_m = 1$
$\delta_M = 2$