Control of Robotic Manipulators

Adaptive Control

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MMAE 540: Introduction to Robotics
What might be the problem with controlling this robot?
Control of Robotic Manipulators

Adaptive Control
1-D Example

• Equations of motion

\[ J\ddot{q} + b\dot{q} |\dot{q}| + mgl \sin(q) = \tau \]

• Two states \( q \) and \( \dot{q} \).
• Define:

\[ \ddot{q}(t) = q(t) - q_d(t) \]
\[ \dot{q}(t) = \dot{q}(t) - \dot{q}_d(t) \]

• Want:

\[ \ddot{q}(t) \to 0 \text{ and } \dot{q}(t) \to 0. \]

• But: \( J, b, m, g, \) and \( l \) are unknown!

• Make things easier, combine to unknown to \( mgl \)
1-D Example Continued

- Unknown vector: \[ \mathbf{a} = \begin{bmatrix} J & b & mgl \end{bmatrix}^T \]

- Introduce a “sliding variable”
  \[ s = \dot{q} + \lambda \ddot{q} \]
  where \( \lambda > 0 \)

- \( s \) is a weighted sum of the position error and the velocity error

- For tracking to be achievable using a finite control signal, the initial desired state must be
  \[ \mathbf{q}_d(0) = \mathbf{q}(0) \]

- Why?

- In a 2\textsuperscript{nd} order system the position or velocity cannot “jump”
  - Any desired trajectory feasible from time \( t=0 \) must start with the same position and velocity of the plant
Given the initial condition $q_d(0) = q(0)$

The tracking problem is equivalent to that of remaining on the surface, $s(t)$ for all $t > 0$.

Problem of tracking a n-dimensional vector $q_d$ is replaced by a 1$\text{st}$ order stabilization problem in $s$

Thus $s \to 0$

Implies that $\dot{q} = -\lambda \ddot{q}$

Implies that $\dot{q} \to 0$ and $\ddot{q} \to 0$

Why? Because those are the only possible solutions

Easier to prove $s \to 0$ than to prove two things approach 0
Example Continued

• Introduce new variable \( \dot{q}_r = \dot{q}_d + \lambda \ddot{q} \)

• So that:
  \[
  s = \dot{q} - \dot{q}_d + \lambda \ddot{q} \\
  s = \dot{q} - \dot{q}_r
  \]

• Choose a Lyapunov function:
  \[
  V = \frac{1}{2} J s^2 \\
  \dot{V} = s J \dot{s}
  \]

• Substitute the system dynamics:
  \[
  \dot{V} = s J \dot{s} \\
  = s J (\dddot{q} - \dddot{q}_r) \\
  = s (\tau - J \dddot{q}_r - b \dddot{q} |\dddot{q}| - mgl \sin q) \\
  = s \left( \tau - \begin{bmatrix} \dddot{q}_r & \dddot{q} |\dddot{q}| & \sin q \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \\
  = s (\tau - Ya)
  \]
Mini-quiz

• Why does proving that $s$ approaches 0 imply that the system is asymptotically stable?

• What does the $\mathbf{a}$ vector represent?

• What does the $\mathbf{Y}$ vector represent?
Example Continued

\[ \dot{V} = s(\tau - Ya) \]

- If \( a \) is known, then choose: \( \tau = Ya - ks \)
- Yields: \[ \dot{V} = s(Ya - ks - Ya) = -ks^2 \]
- Negative definite
  - Globally asymptotically stable!
  - Not quite yet...
  - Why can’t we just stop here?
- Globally asymptotically stable = converge from anywhere to the \textit{equilibrium state}
- What is the equilibrium state of \( s \)?
- Need to show that \( s \to 0 \) as \( t \to \infty \)
Set point vs trajectory control

• In previous system joint velocity approached 0 because that was the goal
  • Set point trajectory
• Can’t assume that anymore because:
  • Trajectory control
  • Instead show that $\dot{q} \rightarrow q_d$
• To do this, show that $\dot{V} \rightarrow 0$ as $t \rightarrow \infty$
• Can’t use Lasalle’s theorem b/c system is not autonomous
• Need Barbalat’s Lemma
Barbalat’s Lemma

- Given a function $f(t)$ that is:
  - differentiable
  - has a limit $f(t) \to L$
  - has a time derivative that is uniformly continuous (i.e. $\ddot{f}$ exists and is bounded)

- Then:
  - $\dot{f}(t) \to 0$ as $t \to \infty$
Barbalat’s Lemma Continued

• Why do we need the third condition (uniform continuity)?
• Because $f(t)$ simply having a limit does not imply that $\dot{f}(t) \to 0$
• Ex:
  
  $$f(t) = e^{-t} \sin (e^{2t})$$

• Sinusoidal decrescence to zero in an exponential envelope
• $\dot{f}$ is actually infinite (i.e. faster and faster oscillations)

• If $\ddot{f}$ exists and is bounded, then $\dot{f}$ is uniformly continuous
Lyapunov-like Lemma that builds off of Barbalat’s lemma

- If $V(x, t)$ has the following properties
  - $V(x, t)$ is lower bounded
  - $\dot{V}(x, t)$ is negative semi-definite
  - $\dot{V}(x, t)$ is uniformly continuous ($\dot{V}(x, t)$ is bounded)
- Then $\dot{V}(x, t) \to 0$ as $t \to \infty$
• For now we are assuming we know the $a$ parameters

\[ \dot{V} = -ks^2 \]

\[ \ddot{V} = -2ks \dot{s} \]

• Compute

• Is that bounded?

• A function is bounded if there exists some number, $M$, such that

\[ |f(x)| \leq M \]

• For all $x$ in some set $X$
Pendulum Example Continued

• So is: \[ \ddot{V} = -2k s \dot{s} \]
  \[ = -2k (\dot{q} + \lambda \ddot{q}) (\ddot{q} + \lambda \ddot{q}) \]

• Bounded?

• Yes, because we can choose what \( q_d \) and \( \dot{q}_d \) are!

• As long as we choose them to be bounded, \( s \) and \( \dot{s} \) will be bounded
Mini Quiz

• What is Barbalat’s Lemma?
But what if we don’t know $a$?

- Make an estimate, $\hat{a}$
- Let $\tilde{a} = \hat{a} - a$
- Use control law: $\tau = Y\hat{a} - ks$
- Find derivative: 
  \[ \dot{V} = s(\tau - Ya) = s(-Y\hat{a} - ks - Ya) = -ks^2 + sY\tilde{a} \]
- Not negative definite!
Try again!

- Remember, Lyapunov functions are sufficient, but not necessary.
- Try another Lyapunov function \( V = \frac{1}{2} Js^2 + \frac{1}{2} \tilde{a} P^{-1} \tilde{a} \)
- Where \( P \) is symmetric and constant.
- \( \tilde{a} \) cannot be constant (more later).
- Thus \( \dot{a} = \dot{a}(t) \)
- Yields:

\[
V = \frac{1}{2} Js^2 + \frac{1}{2} \tilde{a} P^{-1} \tilde{a} \\
\dot{V} = s \left( \tau - Y \dot{a} \right) + \dot{a}^T P^{-1} \tilde{a} \\
= -ks^2 + \left( sY + \dot{a}^T P^{-1} \right) \tilde{a}
\]
Continued

\[
\dot{V} = -ks^2 + \left( sY + \hat{a}^T P^{-1} \right) \hat{a}
\]

- We choose what we want \( \hat{a} \) to be!
- Let \( \hat{a} = -PY^T s \)
- Yields:
  \[
  \dot{V} = -ks^2 + \left( sY - (PY^T s)^T P^{-1} \right) \hat{a}
  \]
  \[
  = -ks^2
  \]
- Negative definite + Barbalat’s Lemma = globally asymptotically stable
What Just Happened?

• Used Lyapunov’s method to **find** a control law that would be stable
• Opposed to using Lyapunov’s method to **prove** a known control law is stable
• Best control law for system?
  • Not necessarily
  • \( \tau = Y\hat{a} - ks \)
  
  Estimate of:

\[
Y = \begin{bmatrix}
\dot{q}_r & q & |\dot{q}| & \sin q
\end{bmatrix}
\]

\[
a = \begin{bmatrix}
J & b & mgl
\end{bmatrix}^T
\]

• How to break up \( Y \) and \( a \) can be tricky
  • More than one way
  • \( k \) term represents gain
  • \( \hat{a} = -PY^T \) represents how fast estimate of parameters changes
Apply to n-link Manipulator

\[
H\ddot{q} + C\dot{q} + g = \tau
\]

- Let

\[
s = \dot{q} + \lambda\ddot{q}
\]
\[
\dot{q}_r = \dot{q}_d - \lambda\ddot{q}
\]
\[
\ddot{a}(t) = \dot{a}(t) - a
\]
\[
\dot{a}(t) = \dot{a}(t)
\]

\[
s = \dot{q} - \dot{q}_r
\]

- Choose

\[
V = \frac{1}{2} s^T H s + \frac{1}{2} \ddot{a}^T P^{-1} \ddot{a}
\]
N-link manipulator example continued

\[ V = \frac{1}{2} s^T H s + \frac{1}{2} \tilde{a}^T P^{-1} \tilde{a} \]

- Positive Definite?
- Negative Definite?

\[ \dot{V} = s^T H \dot{s} + \frac{1}{2} s^T \dot{H} s + \frac{1}{2} \tilde{a}^T P^{-1} \tilde{a} \]

\[ = s^T (H \ddot{q} - H \ddot{q}_r) + \frac{1}{2} s^T \dot{H} s + \tilde{a}^T P^{-1} \dot{a} \]

\[ = s^T (\tau - H \ddot{q}_r - C \ddot{q} - g) + \frac{1}{2} s^T \dot{H} s + \tilde{a}^T P^{-1} \dot{a} \]

\[ = s^T (\tau - H \ddot{q}_r - C \ddot{q} - g) + \frac{1}{2} s^T \dot{H} s + \tilde{a}^T P^{-1} \dot{a} \]

Cancel b/c of skew-symmetry
N-link manipulator example continued

\[ \dot{V} = s^T (\tau - H\ddot{q}_r - C\dot{q}_r - g) + \hat{a}^T P^{-1} \hat{a} \]

- Let
  \[ -Ya = -H\ddot{q}_r - C\dot{q}_r - g \]
- Create a control law:
  \[ \tau = Y\hat{a} - K_D s \]
- Let our estimate of \( a \) be:
  \[ \hat{a} = -PY^T s \]
- then
  \[ \dot{V} = s^T (Y\hat{a} - K_D s - Ya) - \hat{a}^T P^{-1} PY^T s \]
  \[ = -s^T K_D s \]
Some practical notes on Homework

- Difficulty lies in determining Y and a
- Long and frustrating
- Write out equation of motion and look for common terms
- Should be able to do it with 5 terms
- More than 5 makes things more difficult
Example: Adding Viscous Friction

\[ H(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D \dot{q} + g(q) = \tau \]

- Could define \( Y_a \) as \( H\ddot{q}_r + C\dot{q}_r + D\dot{q} + g \)
- Stable, but not optimal
- Instead, let: \( D\dot{q} = Ds + D\dot{q}_r \)
- So:
  \[ H(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D \dot{q} + g(q) = \tau - Ds \]
  \[ H\ddot{q}_r + C\dot{q}_r + D\dot{q}_r + g = Y_a \]
- Yields:
  \[ \dot{V} = -s^T (K_D + D) s \]